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Order formation processes of complex systems including different parity order parameters

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HIGHLIGHTS

- Order formation processes are studied including two different order parameters.
- We introduced a set of Langevin equations in order to express the order formation processes.
- The distribution function of our order formation process is expressed in the Fokker–Planck equations.
- The entropy production and the noise dependence of the onset time are obtained by the distribution functions.
- The four body interaction models are quite different from a spin-glass model from the view point of Langevin equations.

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ABSTRACT

In the present study, we focus on the parity of the order parameters and clarify the order formation process in a system including two order parameters. Each order parameter shows different parity under a gauge transformation, namely even and odd order parameters. For example, in a spin-glass model, the even order parameter corresponds to the spin-glass order parameter while the odd one corresponds to the magnetization. We introduce phenomenologically a set of Langevin equations to express the ordering process under a white Gaussian noise. Using two kinds of Fokker–Planck equations, we analyze the order formation process and the entropy production. Furthermore, we show the noise dependence of the onset time.

1. Introduction

The scaling theory of order formation processes was established by one of the authors (M.S.) in 1976 [1,2]. This scaling theory focuses on an order parameter, namely magnetization, and clarifies the process from an unstable state to the final stable state. According to the scaling theory [1,2], the order formation process can be expressed by the asymptotic expansion with respect to the scaling time τ, namely $\tau \propto e^{2\gamma t}$, where t denotes time, and γ a constant real number [1,2]. The scaling theory can be applied to many phenomena including fish schooling [3] and nuclear fission [4]. In parallel to the works on applications, it was repeatedly tested [5–7] using analytic and numerical methods.

Recently, relaxation processes including the order formation become more important in such complex systems as glass-like systems [8–10]. In many complex systems, there exists a highly symmetric structure in an apparently disordered state. For example, in the spin-glass phase, no magnetization appears, while a spin-glass ordered state appears. In the present

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study, we clarify the ordering processes in a system including two order parameters, namely even-parity order-parameters (namely, "even order parameters") and odd-parity order-parameters (namely, "odd order parameters"). This is one of the typical examples in the above complex systems.

At first, we introduce a set of Langevin equations describing our system with even and odd order parameters. Thus we analyze directly the Langevin equations in Section 2. Using the distribution functions, the Langevin equations can be represented by the Fokker–Planck equations. We solve the Fokker–Planck equations and obtain the relevant onset time in our system in Section 3. The time evolution of the entropy is obtained from the distribution function as shown in Section 4. A typical example of the present order formation process appears in the four-body interaction model [9,10] and a spin-glass model. We discuss a simple derivation of this process from the four-body interaction model in Section 5. A summary and discussions are given in Section 6.

2. Langevin equations including even and odd order parameters

2.1. Basic theory of order formation

In this subsection, we make a brief review of the basic theory of the order formation [1,2] using our notation. It may be useful to introduce our Langevin equations. The basic theory [1,2] expresses the phase transition from an initial unstable state to the final stable state. The time dependence of the order parameter $x(t)$ on the double well potential $V(x) = -(\gamma/2)x^2 + (g/4)x^4$ is obtained by the Langevin equation [1,2]

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \gamma x(t) - gx^3(t) + \zeta(t), \tag{1}$$

where the parameters γ and g depend on the temperature T. Especially the parameter γ takes a positive value ($\gamma > 0$) for $T < T_c$, where the temperature T_c denotes the critical temperature. Note that, the noise $\zeta(t)$ is assumed to be a white Gaussian noise:

$$\langle \zeta(t)\zeta(t')\rangle = 2\epsilon\delta(t-t'), \tag{2}$$

where the average $\langle Q(t)\rangle$ denotes the stochastic average of the random parameter $Q(t)$ under the random noise $\zeta(t)$, and the constant value ϵ denotes the strength of the noise.

Using the Langevin equation (1), the fluctuation $\langle x^2\rangle_t$ of the order parameter $x(t)$ is obtained as Refs. [1,2]

$$\langle x^2\rangle_t \simeq \frac{\langle x^2\rangle_\infty}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\xi^2\tau}{1+\xi^2\tau}\mathrm{e}^{-\xi^2/2}\mathrm{d}\xi \tag{3}$$

under the scaling limit [1,2]

$$\epsilon \to 0, \qquad g \to 0, \qquad t \to \infty \quad \text{and} \quad \tau = \text{finite}, \tag{4}$$

where the scaling time τ is defined [1,2] by

$$\tau \equiv \frac{g\epsilon}{\gamma^2}\mathrm{e}^{2\gamma t}. \tag{5}$$

The scaling limit (4) means the limiting-case of the small noise or small nonlinearity. This scaling limit corresponds to the condition which appears in the order formation processes. As shown in Eqs. (3) and (5), the order formation process is scaled exponentially by the scaling time τ. According to this scaling theory, an ordered state appears in the following time scale, namely "onset time" [1,2]:

$$O(\tau) = 1 \Leftrightarrow t_o = \frac{1}{2\gamma}\log\frac{\gamma^2}{g\epsilon}. \tag{6}$$

2.2. Classification of order parameters in terms of symmetry

As discussed in Section 1, in many complex systems, there exists a highly symmetric structure in an apparently disordered state. Then we classify the order parameters from the view point of the parity, that is, one is the odd-parity order-parameter (namely, the "odd order parameter") and another is the even-parity order-parameter (namely, the "even order parameter"). A typical example including these order parameters is shown in four-body interaction models [9,10]. In this example, the magnetization corresponds to the *odd* order parameter, while the spin-pair corresponds to the *even* order parameter. In four-body interaction models, under the gauge transformation $\{S_i\} \to \{-S_i\}$, the magnetization $m \equiv \langle S_i\rangle$ changes into $-m$, while the spin-pair order parameter $\eta \equiv \langle S_iS_j\rangle$ does not change. Then, in general, the even order parameter is denoted by η and the odd order parameter is denoted by m below. Here the two characteristic temperatures T_η and T_m can be defined as the transition temperatures corresponding to the even and odd order parameters, respectively, as shown in Fig. 1. Because the even order parameter η has higher symmetric property, the transition temperature T_η is higher than T_m. In many cases, such as four-body interaction models or some spin-glass models, the lower critical temperature T_m is zero. Then we focus on the phase in the temperatures $T_m < T < T_\eta$. Thus, we may conclude T_c denotes the critical temperature T_c of the system.