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Steady-state traffic flow on a ring road with up- and down-slopes

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HIGHLIGHTS

- Steady-state flow solution on an inhomogeneous ring road is derived.
- The solution only depends on the total number of vehicles on the ring road.
- The solution is exact and stable for the LWR model.
- The solution is approximate and can be unstable for the higher-order model.

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ABSTRACT

Steady-state traffic flow on a ring road with up- and down-slopes is investigated using a semi-discrete model. By exploiting the relations between the semi-discrete and continuum models, a steady-state solution is uniquely determined for a given total number of vehicles on a ring road. The solution is exact and always stable with respect to the first-order continuum model, whereas it is a good approximation with respect to the semi-discrete model provided that the involved equilibrium constant states are linearly stable. In other cases, the instability of one or more equilibria could trigger stop-and-go waves propagating in certain sections of road or throughout the ring road. The indicated results are reasonable and thus physically significant for better understanding of real-world traffic flow on inhomogeneous roads, such as those with junctions or bottlenecks.

1. Introduction

In statistical physics, traffic flow has been viewed as a self-organized critical system [1,2], and typical phases such as free flow, congestion, jamming together, and their transitions have been interpreted by a fundamental diagram (see Ref. [3] and references therein) and the three-phase theory [4,5]. More recently, traffic has been considered a coevolutionary process, and game theory has been used to depict driving behavior [6,7]. As shown with the Biham–Middleton–Levine (BML) model [8], the choice between cooperation and defection exerts significant influence on the evolution of traffic states [6]. These theories

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were developed with large amounts of modeling and simulation assuming homogeneous road conditions (e.g., a single road with a uniform fundamental diagram).

In contrast, the influence of inhomogeneous road conditions (more generally those with bottlenecks or junctions) on traffic states has become a major concern in the development of traffic flow theory (e.g., see Refs. [3,9]). In macroscopic modeling, the Riemann problem with discontinuous fluxes was studied mainly to design a numerical scheme that could be related to the theory of hyperbolic conservation laws with discontinuous fluxes [10–17]. In microscopic car-following modeling, steady-state flow over inhomogeneous sections of road was analytically or numerically studied by assuming or implying that the solution is a piecewise equilibrium constant in the inhomogeneous sections road [18–23].

Such a steady-state solution can be described using the kinetic theory or the theory of hyperbolic conservation laws with discontinuous fluxes [14,21], which is exact and stable for the first-order continuum Lighthill–Whitham–Richards (LWR) model [24,25]. However, it offers only an approximate solution to the car-following model or the higher-order model with relaxation if the relaxation time is sufficiently small; otherwise, it is unnecessarily stable due to the underlying metastability in these models. Nevertheless, this feature has not yet been well recognized or emphasized in the aforementioned studies of the car-following model.

In this context, the present paper addresses the problem using a semi-discrete model, which can be viewed as an extension of, and thus is more general than, the car-following model [26]. We enhance the mathematical analysis of steady-state flow on a ring road with up- and down-slopes, which poses more complexity for the solution. We also emphasize the instability of the solution with physical interpretation through the analysis of all equilibrium states and numerical simulation. Equilibrium traffic flow with an intermediate density value is widely regarded as unstable on a homogeneous ring road, which with oscillations is liable to evolve into stop-and-go waves (e.g., see Refs. [27–33]). Although the ring road discussed herein is much more complicated, and the simulation does not completely agree with the analytical results due to errors in the analysis, the steady-state solution is shown to have a similar tendency. That is, it is highly stable for large or small number of vehicles on the ring road, and is unstable for intermediate number of vehicles. This result is important for understanding the macroscopic properties of traffic flow on an inhomogeneous road.

In Section 2 of this paper, the semi-discrete model and its correlation with the continuum model are discussed, and the linear stability condition for an equilibrium solution state is indicated. In Section 3, the wave pattern at a stationary interface is described using the theory of hyperbolic conservation laws with discontinuous fluxes, which helps to determine the two adjacent equilibrium constant states. Accordingly, the wave types of the steady-state solution on a ring road with up- and down-slopes, which depend on the total number of vehicles on the ring road, are indicated analytically. In Section 4, initial distributions with certain total number of vehicles are shown to converge to or diverge from the corresponding steady-state solutions through numerical simulation, which generally agrees with the analytical results. Concluding remarks are given in Section 5.

2. The semi-discrete model

In the semi-discrete model [26], a moving "particle" in traffic flow could be numbered by m with $x_m(t)$ being its position, and

$$\frac{\mathrm{d}x_m(t)}{\mathrm{d}t} = u_m(t), \quad m = 0, \pm 1, \pm 2, \ldots \tag{1}$$

being its speed, and the acceleration can be defined through

$$\frac{\mathrm{d}}{\mathrm{d}t}[u_m(t) + p(s_m(t))] = \frac{1}{\tau}[u_e(s_m(t)) - u_m(t)]. \tag{2}$$

Here,

$$s_m(t) = [x_{m+1}(t) - x_m(t)]/\Delta M, \quad \text{and} \quad \rho_m(t) \equiv (s_m(t))^{-1}, \tag{3}$$

are the specific volume and the density, respectively, represented by the particle m, ΔM is the mass between the particles m and $m+1$, and $u_e(s)$ and $p(s)$ are the equilibrium velocity–density relationship and the traffic pressure satisfying $u_e'(s) > 0$, and $p'(s) \le 0$. For $\Delta M = 1$, the semi-discrete model of (1) and (2) reduces to a car-following model, in which case each particle can be viewed as a vehicle in that there is just one vehicle between the heads of two adjacent vehicles. The resulting car-following model is essentially the same as that in Refs. [34,31,32].

2.1. Correlation to the continuum model in Lagrange coordinates

Assume that there is no overtake between particles for the division with a sufficiently small increment ΔM, and let M denote the total mass not passing through particle m. This implies that M referring to particle m is independent of time t. Therefore, particle m can be identified by M and an associated variable $A_m(t)$ (e.g., the position $x_m(t)$ and speed $v_m(t)$) can be re-denoted by $A(M, t)$. Let $\Delta M \to 0$, then the flow can be viewed as a continuum in which M and t constitute the Lagrange coordinate system to describe the variable $A(M, t)$. In this case, Eq. (3) gives

$$s(M, t) = x_M(M, t), \quad \text{and} \quad \rho(M, t) = (s(M, t))^{-1}. \tag{4}$$