Physica A 388 (2009) 1682–1686

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

The Korteweg-de Vries soliton in the lattice hydrodynamic model

H.X. Ge *

Faculty of Science, Ningbo University, Ningbo 315211, China

ARTICLE INFO

Article history:
Received 31 August 2008
Received in revised form 21 October 2008
Available online 27 November 2008

PACS:
05.70.Fh
05.70.Jk
64.60.Fr

Keywords:
Traffic flow
Lattice hydrodynamic model
KdV equation

ABSTRACT

The lattice hydrodynamic model is not only a simplified version of the macroscopic hydrodynamic model, but is also closely connected with the microscopic car following model. The modified Korteweg-de Vries (mKdV) equation about the density wave in congested traffic has been derived near the critical point since Nagatani first proposed it. But the Korteweg-de Vries (KdV) equation near the neutral stability line has not been studied, which has been investigated in detail in the car following model. So we devote ourselves to obtaining the KdV equation from the lattice hydrodynamic model and obtaining the KdV soliton solution describing the traffic jam. Numerical simulation is conducted, to demonstrate the nonlinear analysis result.

1. Introduction

Generally speaking, traffic flow is divided into three distinct regions: the stable region out of the coexisting curve, the metastable region between the coexisting and neutral stability lines, and the unstable region within the neutral stability line. The Burgers, KdV and mKdV equations describe the density waves appearing in the three distinct regions respectively, which are studied in full according to the microscopic car following models [1–7].

Recently, Nagatani [8,9] proposed the lattice hydrodynamic model. It is a simplified version of the macroscopic hydrodynamic model, and also incorporates the idea of the microscopic optimal velocity model. The governing equations are described as

$$\partial_t \rho + \rho_0 \partial_x(\rho v) = 0, \tag{1}$$

$$\partial_t(\rho v) = a\rho_0 V(\rho(x+\delta)) - a\rho v, \tag{2}$$

where ρ_0 is the average density, and a is the sensitivity of a driver; $\rho(x+\delta)$ is the local density at position $x+\delta$ at time t; δ represents the average headway, which means $\delta = 1/\rho_0$; Local density $\rho(x+\delta)$ is related to the inverse of headway $h(x,t) : \rho(x+\delta) = 1/h(x,t)$. The right-hand side of Eq. (2) expresses the tendency of traffic flow ρv at a given density to relax to some natural average flow $\rho_0 V(\rho(x+\delta))$, which is similar to the optimal velocity model proposed by Bando [10]

$$\frac{\mathrm{d}^2 x_j(t)}{\mathrm{d}t^2} = a\left[\tilde{V}(\Delta x_j(t)) - \frac{\mathrm{d}x_j(t)}{\mathrm{d}t}\right]. \tag{3}$$

The idea is that a driver adjusts the car velocity according to the observed headway $\Delta x_j(t)$, which corresponds to the inverse of the local density $h(x,t)$.

* Tel.: +86 574 81953825.
E-mail address: gehongxia@nbu.edu.cn.

0378-4371/$ – see front matter © 2009 Published by Elsevier B.V.
doi:10.1016/j.physa.2008.11.026

The continuity equation (2) is modified with dimensionless space x. Let $\tilde{x} = x/\delta$, and $\tilde{x}$ is indicated as x hereafter. As we know, the lattice hydrodynamic model is presented as three types. Model A is similar to the second order differential equation in the car following model proposed by Bando [10]:

$$\partial_t \rho_j + \rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = 0, \tag{4}$$

$$\partial_t(\rho_j v_j) = a\rho_0 V(\rho_{j+1}) - a\rho_j v_j, \tag{5}$$

and Model B is similar to the first order differential equation in the car following model given by Newell [11]. Including Eq. (4), the current equation is

$$\rho_j(t+\tau)v_j(t+\tau) = \rho_0 V(\rho_{j+1}). \tag{6}$$

Model C is the difference form of the model B, which is similar to that in the car following model put forward by Nagatani [12]. The system consists of Eq. (6) and the following continuity equation

$$\rho_j(t+\tau) - \rho_j(t) + \tau\rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = 0 \tag{7}$$

where the delay time τ is the inverse of the sensitivity a, j denotes site j on the one-dimensional lattice, and $\rho_j(t)$, $vj(t)$ represent the local density and the local average velocity on site j at time t respectively.

The car following models [2,3] and the lattice hydrodynamic models [15,16] have many similar properties, such as that the stable region could be enlarged by taking into account the next-nearest-neighbor interaction [2,15], and based on the intelligent transport system, only the information of three vehicles/sites ahead is enough for cooperative driving [3,16]. In the aspect of the density wave, the microscopic car following model has been studied extensively and thoroughly according to the KdV equation, the Burgers equation and the mKdV equation. But for the macroscopic lattice hydrodynamic model, only mKdV equation was investigated frequently [8,9,15–17], and the evolution of the density wave was not carried out by numerical simulation.

In this paper, based on the original lattice hydrodynamic model A, B and C, we make the conclusion as to the three types of models, through linear and nonlinear methods. The KdV equations are derived near the neutral stability lines by using the reductive perturbation method, and the corresponding soliton solutions describing the density waves are obtained. Numerical simulation is carried out to validate the nonlinear result, which is not given before.

2. Linear stability analysis

The linear stability analyses are made for the three types of lattice hydrodynamic models. It is obvious that the uniform traffic flow with constant density ρ_0 and constant optimal velocity $V(\rho_0)$ is the steady state solution for the three models, given as

$$\rho_j(t) = \rho_0, \qquad v_j(t) = V(\rho_0). \tag{8}$$

Suppose $y_j(t)$ to be a small deviation from the steady state density of the jth vehicle

$$\rho_j(t) = \rho_0 + y_j(t). \tag{9}$$

Eliminating the velocity from the three systems (Eq. (4)–(7)), which lead to

$$\partial_t^2 \rho_j(t) + a\partial_t \rho_j(t) + a\rho_0^2 \left[V(\rho_{j+1}(t)) - V(\rho_j(t))\right] = 0, \tag{10}$$

$$\partial_t \rho_j(t+\tau) + \rho_0^2 \left[V(\rho_{j+1}(t)) - V(\rho_j(t))\right] = 0, \tag{11}$$

$$\rho_j(t+2\tau) - \rho_j(t+\tau) + \tau\rho_0^2 \left[V(\rho_{j+1}(t)) - V(\rho_j(t))\right] = 0. \tag{12}$$

Substituting Eqs. (8)–(9) into Eqs. (10)–(12) and linearizing it yield

$$\partial_t^2 y_j(t) + a\partial_t y_j(t) + a\rho_0^2 V'(\rho_0)\Delta y_j(t) = 0, \tag{13}$$

$$\partial_t y_j(t+\tau) + \rho_0^2 V'(\rho_0)\Delta y_j(t) = 0, \tag{14}$$

$$y_j(t+2\tau) - y_j(t+\tau) + \tau\rho_0^2 V'(\rho_0)\Delta y_j(t) = 0, \tag{15}$$

where $V'(\rho_0) = \mathrm{d}V(\rho_j)/\mathrm{d}\rho_j|_{\rho_j=\rho_0}$, and $\Delta y_j(t) = y_{j+1}(t) - y_j(t)$. Expanding y_j in the Fourier-modes: $y_j(t) = \exp(\mathrm{i}kj + zt)$, we have

$$z^2 + az + a\rho_0^2 V'(\rho_0)(\mathrm{e}^{\mathrm{i}k} - 1) = 0, \tag{16}$$

$$z\mathrm{e}^{z\tau} + \rho_0^2 V'(\rho_0)(\mathrm{e}^{\mathrm{i}k} - 1) = 0, \tag{17}$$

$$\mathrm{e}^{2z\tau} - \mathrm{e}^{z\tau} + \tau\rho_0^2 V'(\rho_0)(\mathrm{e}^{\mathrm{i}k} - 1) = 0. \tag{18}$$

Expanding $z = z_1(\mathrm{i}k) + z_2(\mathrm{i}k)^2 + \cdots$ and inserting it into Eqs. (16)–(18), the first- and second-order terms of $\mathrm{i}k$ are obtained respectively.