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# Hierarchical coefficient of a multifractal based network

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## HIGHLIGHTS

- There are many classes of networks in the market, each one with their peculiarities.
- We work with the Lucena network—the dual of a multifractal lattice.
- The multifractal is by construction a proper partition of a square according to vertical and horizontal sections.
- The Lucena network shows the hierarchical property, a power-law relation between clustering coefficient and connectivity.
- We work a mathematical demonstration connecting clustering coefficient and connectivity for any scale-free planar network.

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ABSTRACT

The hierarchical property for a general class of networks stands for a power-law relation between clustering coefficient, $CC$ and connectivity $k$: $CC \propto k^{\beta}$. This relation is empirically verified in several biologic and social networks, as well as in random and deterministic network models, in special for hierarchical networks. In this work we show that the hierarchical property is also present in a Lucena network. To create a Lucena network we use the dual of a multifractal lattice **ML**, the vertices are the sites of the **ML** and links are established between neighbouring lattices, therefore this network is space filling and planar. Besides a Lucena network shows a scale-free distribution of connectivity. We deduce a relation for the maximal local clustering coefficient $CC_i^{max}$ of a vertex $i$ in a planar graph. This condition expresses that the number of links among neighbour, $N_{\triangle}$, of a vertex $i$ is equal to its connectivity $k_i$, that means: $N_{\triangle} = k_i$. The Lucena network fulfils the condition $N_{\triangle} \simeq k_i$ independent of $k_i$ and the anisotropy of **ML**. In addition, $CC^{max}$ implies the threshold $\beta = 1$ for the hierarchical property for any scale-free planar network.



## 1. Introduction

Hierarchical networks are a class of graphs formed by the successive assembling of a same pattern or replication factor [1]. This constructive idea resembles the concept of a fractal but it is not properly the case as pointed in Ref. [2] that has the suggestive title "Pseudo fractal scale-free web". Curiously, the articles that introduced hierarchical networks have not used the hierarchical concept but instead they have explored the deterministic construction process: [2,3]. Indeed, the main appeal of hierarchical networks does not come from their formation algorithm but from a peculiar property that arises from its clustering coefficient distribution.

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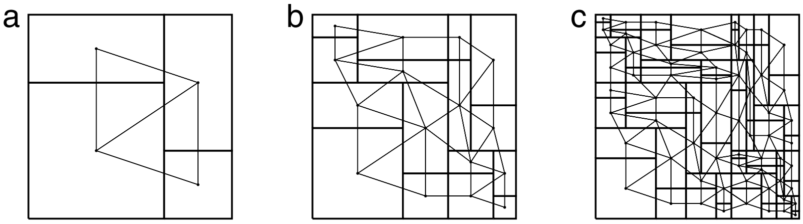


**Fig. 1.** The algorithm of Lucena multifractal lattice and network. The thick lines represent the initial square lattice and the vertical and horizontal sections according to the parameter $\rho = 1/2$ (the blocks are divided into $3 = 1 + 2$ segments and the section splits into two parts of length 1 and 2). The thin lines represent the links connecting neighbour sites. Figures (a)–(c) represent steps $n = 1$, 2 and 3 respectively. In the passage from $n$ to $n + 1$ each vertex is erased and replaced by four new ones; the number of new connections formed at each step is not trivial. Most junctions among the blocks have a T-like shape ($\top, \vdash, \perp$ or $\dashv$), but some are cross-like ($+$). The cross-like intersections are related to topological defects.

The small world network as described in Ref. [4] has large average clustering coefficient, $\langle CC \rangle$, and this characteristic was very important in the beginning of network studies to differentiate it from random networks [5]. However, once we compute the distribution of the cluster coefficient according to connectivity $CC(k)$ the small world network reveals a plain distribution, that means, the local clustering coefficient of each vertex $V_i$, $CC_i$ does not vary with connectivity. A set of challenging empirical results appears when $CC(k)$ was computed for cell metabolic networks [1,6], semantic web [7,8] and the World Wide Web [9,7]. All these networks follow the distinctive pattern:

$$CC(k) \propto k^{-\beta}. \tag{1}$$

In addition, the hierarchical network, that is a theoretical model, also presents the same power-law curve, a fact that attracted the attention of the scientific community to this special type of network. Because of Ref. [7] the coefficient $\beta$ was called the hierarchical coefficient, and Eq. (1) the hierarchical property.

On the other hand, hierarchical property (1) does not necessarily imply a hierarchical structure, we cite for instance the scale-free algorithms [10,11] that are not hierarchical. The work [10] proposed a simple and direct model to generate scale-free networks that also shows the clustering property; the model is similar to the usual preferential attachment model, but it introduces an additional step that increases the average clustering coefficient by choosing additional attachment among highly connected vertices.

The paper [12] introduced a partition of the square that forms a multifractal tilling, that means, the area lattice follows a multifractal distribution. A nice review of multifractal lattice properties is [13]. In that paper the multifractal lattice is called a Lucena multifractal in contrast to another bidimensional multifractal object developed in Ref. [14]. In this work we follow the same line and call the Lucena network the graph using the connectivities (neighbourhood) of lattice elements; in this way the Lucena network is the dual of the multifractal lattice. We avoid the term multifractal network because the area of lattice size is multifractal, however the network formed from the lattice topology is not multifractal.

This paper shows that the Lucena network shares the same basic property of hierarchical networks—the power law relation between clustering coefficient and connectivity with $\beta = 1$. In addition, we present evidence that any scale-free planar (non-crossing network) should also obey the hierarchical property. In Section 2 we review the main properties of a multifractal lattice and explore its basic network characteristics. In Section 3 we present the curves of the clustering coefficient against connectivity for several values of parameter $\rho$ which define the multifractal; in addition a discussion about $CC_i$ of bidimensional non-crossing networks is presented. In Section 4 we conclude the work and discuss similarities between Lucena, Apollonius and hierarchical networks.

## 2. Construction of the multifractal based network

To show the algorithm of Lucena multifractal we start with a square of size 1 and a given partition parameter $0 < \rho < 1$. For convenience we define $\rho = s/r$ for $s$ and $r$ integers which means that each line segment is divided in $r+s$ equal segments which in the sequence are split in two parts of sizes $r$ and $s$. The first step, $n = 1$, consists of two sections of the square: a vertical and a horizontal both following the same $\rho$. In this way the initial square of area one is divided in four blocks of areas: $\rho^2$, $(1 - \rho)^2$, and two of $\rho(1 - \rho)$. The second step repeats the same procedure inside each of the 4 blocks. Using this precedence at step $n$ there are $2^{2n}$ tiles. A simple picture of this algorithm is shown in Fig. 1 for $n = 1$, 2 and 3. A more involved discussion about the construction of this object is found in Refs. [13,15] the possible rotation of the section cut and, also, lattices with random $\rho$ are discussed.

In the Lucena partition of the square the difference between the largest and the smallest lattice areas increases as $\rho \to 0$, moreover, some lattice elements get more and more stretched in this limit; in this way, $\rho$ is a measure of anisotropy. In the opposite limit, $\rho \to 1$, the partition degenerates into the regular square lattice, a very symmetric object. In Fig. 2 we show two realisations of Lucena network for $\rho = 1/3$ (a) and $\rho = 4/5$ (b). The strong asymmetry of (a), small $\rho$, contrasts with the more balanced lattice area and neighbour connectivity of (b), $\rho$ close to one.

The most important property of the Lucena multifractal is that the cutting process segments the initial square into blocks whose area distribution follows a binomial distribution [16]. From this distribution it is possible to calculate the spectrum of