Physica A 438 (2015) 210–222

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Anomalous diffusion in nonhomogeneous media: Power spectral density of signals generated by time-subordinated nonlinear Langevin equations

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HIGHLIGHTS

- We obtained time-subordinated nonlinear SDEs generating signals with power law distributions.
- The analytical expression of power spectral density has been derived.
- For some parameters our equations generate signals having $1/f$ spectrum.

ARTICLE INFO

Article history:
Received 1 May 2015
Received in revised form 10 June 2015
Available online 8 July 2015

Keywords:
Fractional Fokker–Planck equation
Stochastic analysis methods
Systems obeying scaling laws
$1/f$ noise
Power law tails

ABSTRACT

Subdiffusive behavior of one-dimensional stochastic systems can be described by time-subordinated Langevin equations. The corresponding probability density satisfies the time-fractional Fokker–Planck equations. In the homogeneous systems the power spectral density of the signals generated by such Langevin equations has power-law dependency on the frequency with the exponent smaller than 1. In this paper we consider nonhomogeneous systems and show that in such systems the power spectral density can have power-law behavior with the exponent equal to or larger than 1 in a wide range of intermediate frequencies.

1. Introduction

A number of experimental observations show that more complex diffusion processes in which the mean-square displacement is not proportional to the time t take place in various systems. A broad family of processes described by certain deviations from the classical Brownian linear time dependence of the centered second moment is called anomalous diffusion. Anomalous diffusion in one dimension is characterized by the occurrence of a mean square displacement of the form

$$\langle (\Delta x)^2 \rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha, \qquad (1)$$

which deviates from the linear Brownian dependence on time [1]. Eq. (1) introduces the anomalous diffusion coefficient K_α. Such a deviation from classical diffusive behavior can be observed in many systems [2–4] and leads to many interesting physical properties [5]. Applications of anomalous diffusion have been found in physics, chemistry and biology [1,5,6]. In general, anomalous diffusion occurs in complex structures exhibiting the presence of long-range correlations or memory

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http://dx.doi.org/10.1016/j.physa.2015.06.047

effects [1]. In the physics of complex systems, anomalous transport properties and their description have attracted considerable interest starting with the pioneering papers of Montroll and his collaborators [7].

An important subclass of anomalous diffusion processes constitute subdiffusion processes, characterized by the sublinear dependence with the power-law exponent in the range $0 < \alpha < 1$. In this situation no finite mean jump time Δt exists [2]. Subdiffusion processes have been reported in condensed matter systems [2], ecology [3], and biology [4]. Continuous time random walks (CTRWs) with on-site waiting-time distributions falling slowly as $t^{-\alpha-1}$ and lacking the first moment predict a subdiffusive behavior and are powerful tools to describe systems which display subdiffusion [2,8]. Starting from the generalized master equation or from the CTRW the fractional Fokker–Planck equation can be rigorously derived [9,10]. Fractional Fokker–Planck equation provides a useful approach for the description of transport dynamics in complex systems which are governed by anomalous diffusion [2] and nonexponential relaxation patterns [11]. It has been used to model dynamics of protein systems and for reactions occurring in disordered media [2,12–18]. Description equivalent to a fractional Fokker–Planck equation consists of a Markovian dynamics governed by an ordinary Langevin equation but proceeding in an auxiliary, operational time instead of the physical time [19]. This Markovian process is subordinated to the process defining the physical time; the subordinator introduces memory effects [20]. Other approaches for the theoretical description of the subdiffusion use the generalized Langevin equation [21–23], fractional Brownian motion [24], or the Langevin equation with multiplicative noise [25].

The traditional CTRW provides a homogeneous description of the medium. More complex situation is the diffusion in nonhomogeneous media, for example diffusion on fractals and multifractals [26]. Nonhomogeneous systems exhibit not only subdiffusion related to traps, but also enhanced diffusion can occur: for example, transport of interacting particles in a weakly disordered media is superdiffusive due to the disorder and subdiffusive without the disorder [27]. Anomalous diffusion in heterogeneous fractal medium has been considered in Ref. [28] where it was proposed that in one dimension the mean square displacement has the form $\langle(\Delta x)^2\rangle \sim x^{-\theta}t^{\alpha}$ instead of Eq. (1). Heterogeneous fractional Fokker–Planck equation on heterogeneous fractal structure media has been investigated in Refs. [29–32]. In nonhomogeneous media the properties of a trap can reflect the medium structure, therefore in the description of transport in such a medium the waiting time should explicitly depend on the position. This dependence can be introduced by using the position-dependent subdiffusion exponents [33–35]. Another way is to consider position-dependent time subordinator [36].

In the homogeneous systems the power spectral density (PSD) of the signals generated by time-subordinated Langevin equations has power-law dependency $S(f) \sim f^{\alpha-1}$ on the frequency as $f \to 0$ [37]. Since $0 < \alpha < 1$, the power-law exponent $1-\alpha$ is smaller than 1. The purpose of this paper is to consider the PSD in nonhomogeneous systems exhibiting anomalous diffusion. We demonstrate, that in such systems the PSD can have power-law behavior with the exponent equal to or larger than 1 in a wide range of intermediate frequencies.

The paper is organized as follows: in Section 2 we introduce the time-fractional Fokker–Planck equation describing subdiffusion in nonhomogeneous media. The expression for the power spectral density of the fluctuations of the diffusing particle in such a medium is obtained in Section 3. In Section 4 we consider a particular case of the time-fractional Fokker–Planck equation involving the coefficients with power-law dependence on the position. Numerical methods of solution are discussed in Section 5. Section 6 summarizes our findings.

2. Time-fractional Fokker–Planck equation for nonhomogeneous media

In this section we derive the time-fractional Fokker–Planck equation describing diffusion of a particle in nonhomogeneous media. Usually the description of the anomalous diffusion is given by the CTRW theory assuming heavy-tailed waiting-time distributions between successive jumps of the diffusing particle. Here we use the method of the derivation that is similar to that outlined in Refs. [19,38]. We start with the Markovian process described by the Itô stochastic differential equation (SDE)

$$\mathrm{d}x(\tau) = a(x(\tau))\mathrm{d}\tau + b(x(\tau))\mathrm{d}W(\tau). \tag{2}$$

Here $W(\tau)$ is the standard Brownian motion (Wiener process). The drift coefficient $a(x)$ and the diffusion coefficient $b(x)$ explicitly depend on the particle position x. This dependence on the position reflects the nonhomogeneity of a medium. Following Ref. [19] we interpret the time τ in Eq. (2) as an internal, operational time. Eq. (2) we consider together with an additional equation that relates the operational time τ to the physical time t. The difference between physical time t and the operational time τ occurs due to trapping of the diffusing particle. For the trapping processes that have distribution of the trapping times with power law tails, the physical time $t = T(\tau)$ is given by the strictly increasing α-stable Lévy motion defined by the Laplace transform

$$\langle \mathrm{e}^{-kT(\tau)}\rangle = \mathrm{e}^{-\tau k^{\alpha}}. \tag{3}$$

Here the parameter α takes the values from the interval $0 < \alpha < 1$. Thus the physical time t obeys the SDE

$$\mathrm{d}t(\tau) = \mathrm{d}L^{\alpha}(\tau), \tag{4}$$

where $\mathrm{d}L^{\alpha}(\tau)$ stands for the increments of the strictly increasing α-stable Lévy motion $L^{\alpha}(\tau)$. For such physical time t the operational time τ is related to the physical time t via the inverse α-stable subordinator [39,40]

$$S(t) = \inf\{\tau : T(\tau) > t\}. \tag{5}$$