Physica A 438 (2015) 493–508

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Exact cumulant Kramers–Moyal-like expansion

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HIGHLIGHTS

- Being an exact approach to derive a time-evolution equation for the PDF of a generic system.
- Having a distinct structure of jump-moments from the usual Fokker–Planck or Kramers–Moyal equations.
- Furnishing the time-evolution equation for systems that are not necessarily driven by Langevin-like equations.
- Allowing us to obtain the exact evolution for all the averages and cumulants of the system.

ARTICLE INFO

Article history:
Received 23 February 2015
Received in revised form 18 June 2015
Available online 17 July 2015

Keywords:
Kramers–Moyal Equation
Cumulants

ABSTRACT

We derive an exact equation, a Cumulant Kramers–Moyal Equation (CKME), quite similar to the Kramers–Moyal Equation (KME), for the probability distribution of a Markovian dynamical system. It can be applied to any well behaved (converging cumulants) continuous time systems, such as Langevin equations or other models. An interesting but significant difference with respect to the KME is that their jump-moments are proportional to cumulants of the dynamical variables, but not proportional to central moments, as is the case for the KME. In fact, they still obey a weaker version of Pawula's theorem, namely Marcinkiewicz's theorem. We compare the results derived from the equations herein with the ones obtained by computing via Gaussian and biased, and unbiased, Poisson Langevin dynamics and a Poisson non-Langevin model. We obtain the exact CKME time-evolution equation for the systems, and in several cases, those are distinct from the Fokker–Planck equation or the KME.

1. Introduction

The emergence of macroscopic behavior for a system, out of its microscopic degrees of freedom, is one of the most fascinating aspects of physics [1]. The emergent collective behavior presents many universal properties that are not easily guessed from the microscopic components [2–4].

Indeed, the number of microscopic variables is in general so large that it becomes a useless exercise to try and solve the problem of a macroscopic system from first principles. A more useful approach is to reduce the number of variables describing the system. For instance, this can be done when there is a clear time-scale separation that allows us to take averages over the fast-scale variables and then write equations of motion for the slow-scale variables [5,6]. The main goal is that by reducing the number of variables we can still describe the principal aspects of the system and solve it by analytical or numerical means. Let us call this the macroscopic approach.

The basic approaches in describing a system with macroscopic equations are: via averages (as in thermodynamics or hydrodynamics [7]); via probability distributions (as in the Fokker–Planck and other master equations [8]); and via effective equations of motion for selected variables (such as the Langevin equation [9]).

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http://dx.doi.org/10.1016/j.physa.2015.07.012
0378-4371/

Let us start from a fundamental microscopic equation, such as Liouville's or Schroedinger's Equation, aiming to reduce its variables to a manageable reduced set. The chosen set must have properties that make it suitable for describing the macroscopic properties of the system. For instance, the hydrodynamic equations are derived for the conserved quantities of the system (mass, momentum and energy) [10]. When a subset of the microscopic variables varies considerably slower than the rest, such as the variables describing a Brownian particle immersed in a thermal bath of lighter particles [11], we can obtain time-evolution equations for the variables describing the Brownian particle by taking averages over the variables of the thermal bath: for the Brownian variables (Langevin equation approach) or a Fokker–Planck Equation (FPE) describing the time-evolution of the probability distribution. Usually, the methods above involve truncating a series on powers of a small dimensionless parameter [12], such as the mass ratio for the bath particle mass over the Brownian particle mass. Whereas the FPE is a second order differential equation there is another equation, namely the Kramers–Moyal Equation [13,14] (KME) that is an infinite order differential equation. In fact, the FPE is just the KME truncated up to second order. Here, we are mainly interested in obtaining a KME-like equation that is completely equivalent to the full dynamics of the system but presents distinct properties for its coefficients than those for the usual KME.

Specifically, we develop a method for obtaining the time-evolution equation for the probability distribution for a general dynamical system, we shall call that equation the Cumulant Kramers–Moyal Equation (CKME). As long as the convergence conditions are obeyed (convergence of cumulants) the detailed character of the system does not affect the final form for that equation.

Some specific examples shall be worked out in the context of Brownian motion. Our approach is generic in principle, but we chose to illustrate the procedures by the use of linear Markovian mechanical models for Gaussian, and non-Gaussian noise. We use a formalism based on the Laplace transform that has been used to obtain exact results for Langevin equations in the context of Markovian and non-Markovian noise [15,16], thermal conductance for linear [17] and non-linear systems [18], fluctuation relations for the work [19], and non Gaussian noises [20,21,18,22]. This method is equivalent to taking into account all noise cumulants, which can be of finite or infinite in number. We also exploit a simple non-Langevin Poisson model and obtain the exact CKME which is neither FPE nor KME.

This paper is organized as follows. In Section 2, we discuss the FPE, and some of its properties. In Section 3 we discuss the Kramers–Moyal Equation (KME) and some of its constraints, such as Pawula's theorem. In Section 4, we obtain a Laplace transformed form for the probability distribution function for a generic well behaved Markovian system. In Section 5, we derive a general Cumulant Kramers–Moyal Equation (CKME) for that Markovian system. In Section 6, we discuss some similarities and dissimilarities between the KME and the CKME. In Section 7, we derive the CKME for Gaussian Langevin systems. In Section 8, we derive it for biased and unbiased Langevin Poisson models. In Section 9, we discuss a simple non-Langevin Poisson model. In Section 10, we discuss the results.

2. The Fokker–Planck equation

The Fokker–Planck Equation (FPE) is a staple of statistical mechanics. Technically, it is a continuous master equation, a second order differential equation that can be applied to many situations of real interest, ranging from finance to plasma physics. It reads in general (x represents the set of variables, such as positions y and velocities v)

$$\partial_t p(x,t) = -\frac{\partial}{\partial x}\left[a^{FP}_{(1)}(x)\;\; p(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[a^{FP}_{(2)}(x)\;\; p(x,t)\right], \tag{1}$$

where the jump moments (JM) $a^{FP}_{(n)}(x)$ are defined as the central moments [23] below

$$a^{FP}_{(n)}(x) = \lim_{\tau\to 0}\int_{-\infty}^{\infty} \mathrm{d}x'\,\frac{(x'-x)^n}{\tau}\,p(x', t+\tau \mid x, t). \tag{2}$$

The JM can be experimentally determined by either measuring, or calculating, the averages below [23]

$$a_1 = \frac{\langle \Delta x\rangle}{\Delta t}, \quad \text{and} \quad a_2 = \frac{\langle \Delta x^2\rangle}{\Delta t},$$

for small Δt (where $\Delta x = x' - x$).

An important example of a FPE is the Kramers equation [23]. It governs the probability density for a Brownian particle ($x \equiv (y, v)$) under the action of Gaussian noise and reads

$$\partial_t p(y,v,t) = -v\frac{\partial}{\partial y}p(y,v,t) + \frac{\partial}{\partial v}\left[\left(\frac{\gamma\, v - V'(y)}{m}\right)p(y,v,t)\right] + \frac{\gamma\, T}{m^2}\frac{\partial^2}{\partial v^2}p(y,v,t). \tag{3}$$

This is the classical result for the Kramers equation [24] for a particle under a potential $V(y)$. It is straightforward to verify that the equilibrium solution corresponds to the Boltzmann–Gibbs distribution ($k_B = 1$)

$$p_{eq}(y,v) = \frac{\mathrm{e}^{-\frac{mv^2}{2T} - \frac{V(y)}{T}}}{Z}, \quad \text{where } Z = \int \mathrm{d}y\,\mathrm{d}v\,\mathrm{e}^{-\frac{mv^2}{2T} - \frac{V(y)}{T}}. \tag{4}$$