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# Bifurcation analysis of a speed gradient continuum traffic flow model

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## HIGHLIGHTS

- A bifurcation analysis approach is presented on a macroscopic traffic flow model.
- The types and stabilities of the equilibrium solutions of the model are discussed.
- The conditions of the model's Hopf bifurcation and saddle–node bifurcation are deduced.
- Various bifurcations such as Hopf, saddle–node, limit cycle, Cusp and Bogdanov–Takens are found.
- The Hopf bifurcation can help to explain the stop-and-go traffic phenomena.

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## ABSTRACT

A bifurcation analysis approach is presented based on the macroscopic traffic flow model. This method can be used to describe and predict the nonlinear traffic phenomena on the highway from a system global stability perspective. Based on a recently proposed speed gradient continuum traffic flow model, the types and stabilities of the equilibrium solutions are discussed and the existence of Hopf bifurcation and saddle–node bifurcation is proved. Then various bifurcations such as Hopf bifurcation, saddle–node bifurcation, Limit Point bifurcation of cycles, Cusp bifurcation and Bogdanov–Takens bifurcation are found and the traffic flow behaviors at some of them are analyzed. When the Hopf bifurcation is selected as the starting point of density temporal evolution, it may help to explain the stop-and-go traffic phenomena.



## 1. Introduction

The complex nonlinear traffic phenomena have been studied for decades with the aim of ultimately alleviating and preventing traffic jams. A lot of them have been discovered by researchers on highways and urban expressways, such as traffic breakdown, hysteresis effect, synchronized flow, stop-and-go traffic, wide moving jam, traffic bottlenecks, shock wave, and rarefaction wave. Among them, stop-and-go traffic is an important focus of research in this area. In 1971, Payne [1] proposed a high order continuous model to describe the typical stop-and-go phenomenon. Then, B.S. Kerner et al. [2] studied the structure of stop-and go waves and the characteristic parameters of it in 1994. The phenomenon can also be analytically described by studying traveling waves in most of the current higher-order models [1,3,4].

The analysis of traffic phenomena has also gradually become the mainstream of research in the past decades. All kinds of traffic phenomena have been evaluated by means of observation and controlled experiments [5–8]. Paralleled with experiments, many physical models have been proposed to describe the traffic phenomena and optimize traffic flow

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[2,9–12]. The phase-plane analysis has been proven to be a suitable tool for analyzing the complex traffic flow phenomena [4,13,14]. Lee et al. [13] explained seven types of traveling wave based on the so-called inertia car-following model. Saavedra and Velasco [14] adopted a numerical method, and discussed the types and stabilities of the equilibrium solutions with the Kerner–Konhauser (KK) model [15] and Velasco–Marques (VM) model [16]. C.X. Wu et al. [4] used the phase-plane analysis to study the traveling wave solution of a higher-order traffic flow model under the Lagrange coordinate system, and the results were comparable with those of the analysis. In addition, the bifurcation analysis has also got the particular attention by many researchers.

The bifurcation phenomena are one of the major causes of the nonlinear behaviors in the traffic flow. Y. Igarashi et al. [17] and G. Orosz et al. [18] selected different optimal velocity function to discuss the bifurcation phenomena of the OV model and obtained the similar conclusions. Nagatani [19] analyzed the transforms between regular motions, periodic motions and chaotic motions in the traffic flow by numerical simulation and found that the transforms have rich relations with the bifurcation phenomena. Huijberts [20] mainly studied the public traffic system and found the Hopf bifurcation and the kink jam phase when parameters were changed in certain range.

However, these literatures mentioned above all studied the bifurcation phenomena based on car-following models. Few bifurcation analyses about the macroscopic traffic flow models are known. In this paper, we presented a bifurcation analysis approach to study a recently proposed macroscopic traffic flow model [21]. First, we studied the classification and stability of the equilibrium points about the model and drew the overall distribution structure of the nearby equilibrium solutions in the phase plane. So we can analyze all the traffic phenomena from a global stability perspective. Then we proved the existence of Hopf bifurcation and saddle–node bifurcation and found various bifurcations which can change the system stability by numerical simulation. Some of the bifurcations have been reported previously only for microscopic traffic models [17–20] but not been reported for macroscopic hydrodynamic models. When the Hopf bifurcation is selected as the starting point of density temporal evolution, there are density oscillations with equal amplitude and periodicity. It may help to explain the stop-and-go traffic phenomena.

The remainder of the paper is organized as follows. In Section 2, the non-linear and the linearized systems for solving the traveling wave are deduced, and the types and stabilities of the equilibrium solutions in the systems are discussed. Furthermore, the trajectories of the numeral solutions near the equilibrium points are drawn in the phase plane in comparison with the analytical results. In Section 3, we deduced the conditions of the models Hopf bifurcation and saddle–node bifurcation and discussed the models Hopf bifurcation type. In Section 4, we found various bifurcations in the system such as Hopf bifurcation, saddle–node bifurcation, Limit Point bifurcation of cycles, Cusp bifurcation and Bogdanov–Takens bifurcation by numerical simulations and analyzed the nonlinear dynamical behaviors of the traffic flow at some bifurcation critical point, especially the stop-and-go traffic phenomena. We conclude the paper in Section 5.

## 2. Stability of equilibrium points in the model

L.L. Lai et al. [21] proposed a new speed gradient continuum traffic flow model, which consist of the following two equations, an equation for local vehicle number conservation,

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0 \tag{1}$$

and an equation of motion,

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{V_e-v}{T}+c_0\frac{\partial v}{\partial x}+\mu_0\frac{\partial^2 v}{\partial x^2} \tag{2}$$

where $\rho$ is the density; $v$ is the velocity; $x$ and $t$ represent space and time respectively; $T$ is the driver's reaction time; $V_e[\rho(x,t)]$ is the optimal velocity function; $c_0$ is the speed at which a small disturbance in traffic density (or speed) propagates back along a line of vehicles; $\mu_0=\frac{1}{2}\tau {c_0}^2$ depends on the relaxation time.

L.L. Lai et al. [21] studied the model with a periodic boundary condition. In fact, the traffic phenomena induced by the bifurcation critical points on an open road are much more obvious than those on a closed road, so we here assume that the main road is an open road, i.e.,

$$\rho(1,t)=\rho(2,t), \qquad \rho(L,t)=\rho(L-1,t), \qquad v(1,t)=v(2,t), \qquad v(L,t)=v(L-1,t). \tag{3}$$

We transform the reference system of the wave by introducing the coordinate [2].

$$z=x-ct \tag{4}$$

where $c<0$ is the constant velocity of the moving reference frame with respect to the stationary reference frame. In this case Eq. (1) can be immediately integrated in such a way that

$$\rho(v-c)=q_* \tag{5}$$

where $q_*$ is a constant. Substituting Eqs. (4) and (5) into Eq. (2), we obtain

$$\left[\frac{\mu_0(q_*+c\rho)}{\rho}-c\mu_0\right]\rho_{zz}-\left[\frac{(q_*+c\rho)^2}{\rho^2}-\frac{(2c+c_0)(q_*+c\rho)}{\rho}+\left(c^2+cc_0\right)\right]\rho_z+\frac{q_*+c\rho}{T}-\frac{\rho V_e(\rho)}{T}=0. \tag{6}$$