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Probability of multiple crossings and pricing of double barrier options

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ABSTRACT

This paper derives pricing formulas of standard double barrier option, generalized window double barrier option and chained option. Our method is based on probabilitic approach. We derive the probability of multiple crossings of curved barriers for Brownian motion with drift, by repeatedly applying the Girsanov theorem and the reflection principle. The price of a standard double barrier option is presented as an infinite sum that converges very rapidly. Although the price formula of standard double barrier option is the same with Kunitomo and Ikeda (1992), our method gives an intuitive interpretation for each term in the infinite series. From the intuitive interpretation we present the way how to approximate the infinite sum in the pricing formula and an error bound for the given approximation. Guillaume (2003) and Jun and Ku (2013) assumed that barriers are constant to price barrier options. We extend constant barriers of window double barrier option and chained option to curved barriers. By employing multiple crossing probabilities and previous skills we derive closed formula for prices of 16 types of the generalized chained option. Based on our analytic formulas we compute Greeks of chained options directly.

1. Introduction

A barrier option is a path dependent option which is activated or became worthless on whether the underlying asset price touches or crosses a barrier or barriers before an expiration. There are many

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kinds of barrier options. For example, an up-and-out call single barrier option pays off max $\{S_T - K, 0\}$ to the holder unless the underlying asset touches a predetermined upper barrier at any time before a maturity. An investor may buy a double knock-out barrier option if he/she expects that the underlying asset price will remain within two barriers.

Merton (1973) derived a down-and-out call price using partial differential equation. Rubinstein and Reiner (1991) presented pricing formula for all kinds of single barrier options. They assumed that every option has constant barrier since single barrier option with exponential barrier can be priced by the same formula using simple measure change. However this is not applicable to double barrier options with curved barriers.

To obtain pricing formulas of double barrier options, Kunitomo and Ikeda (1992) employed a generalized Lévy formula and found the density function for Brownian motion reaching in an interval at a maturity without hitting either the lower or the upper curved barriers. Geman and Yor (1996) derived the Laplace transform of the double barrier option price, and Pelsser (2000) inverted the Laplace transform of the probability density function into the analytical formula of the double barrier option price using a contour integration. Not only the Laplace transform but also the Wang transform used to price a vanilla option by Hamada and Sherris (2003) and exotic options by Labuschagne and Offwood (2013). Li (1999) derived a formula of a line segment option, and Baldi, Caramellino, and Iovino (1999) developed the 'sharp large deviation' method for improving the Monte Carlo method. Other researchers including Sidenius (1998), Lin (1999) and Buchen and Konstandatos (2009), also derived the standard double barrier option price formula using different methods, path counting, Gerber–Shiu technique, the method of images and so on.

For a given barrier option we obtain an another option by changing a prespecified trigger event or other conditions. By changing a monitoring period we obtain a front-end barrier option, a rear-end barrier option and a window barrier option. Eliminating one barrier from a double barrier option makes a single barrier option. A rolling option has a chain of barriers which are activated under some conditions. Modifying a trigger event in a rolling option gives a chained option.

Heynen and Kat (1995) priced front-end and rear-end double barrier options whose monitoring periods were either early-ending or forward-starting. A window double barrier option is a double barrier option whose monitoring period starts after a contract initiation and terminates before the contract expiry. Armstrong (2001) and Guillaume (2003) found the valuation formula of a window double barrier option whose barriers are constants, and Guillaume (2010) priced a double barrier option whose barriers are step functions.

A rolling option, proposed by Gastineau (1994), has multiple barriers whose levels are monotonic. For example, with given barriers, $S_0 > H_1 > H_2 > \cdots > H_n$, and strikes, $K_0 > K_1 > \cdots > K_{n-1}$, a roll down call is a European call with a strike K_0 at first. If underlying asset crosses the first barrier, H_1, then a roll down call becomes another European call with strike K_1. Whenever next barrier is hit, the strike is changed again and another next barrier is activated. The option is knocked-out if the last barrier, H_n, is hit. If a barrier is activated in opposite side alternatively then it will be chained option.

A chained option is a barrier option whose predetermined event is whether underlying asset price crosses barrier or barriers in a given order before maturity. For example, an up-and-in call option is a chained option with a single barrier and predetermined event which is hitting the upper barrier. UIC_d, a kind of up-and-in call option, is a chained option with double barriers which pays off max $\{S_T - K, 0\}$ to the holder if underlying asset price falls down the lower barrier, then hits the upper barrier before the maturity. Li (1999) presented a new option with this structure as an example of his work. A chained option was independently rediscovered and named as the chained option by Jun and Ku (2013).

The rest of this paper is organized as follows: Section 2 derives the cumulative distribution of multiple crossings of the upper and the lower barriers. Section 3 presents the pricing formula for the standard double barrier option as an infinite sum and explains why its speed of convergence is very fast. Section 4 investigates the window double barrier option with curved barriers. Section 5 derives analytical pricing formulas and Greeks for a generalized chained option for any number of barrier crossings. Section 6 concludes the paper.