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Parameter estimation for the generalized fractional element network Zener model based on the Bayesian method

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HIGHLIGHTS

- The generalized fractional element network Zener model is considered.
- The Bayesian method to estimate parameter for GFE network Zener model is proposed.
- Three examples are performed to prove the validity of the Bayesian method.
- The Bayesian method is shown to be feasible for fractional constitutive equations.

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ABSTRACT

In the present study, an inverse problem to estimate parameters in the Zener model of viscoelasticity based on the generalized fractional element (GFE) network is studied. The Bayesian method is proposed to obtain the optimal estimation of the viscoelastic parameters. Three examples are performed to certify the validity of the method. All numerical results lead to an excellent fitting between the calculative results and experimental data. It is shown that the Bayesian method is feasible in the inverse problem of parameter estimation for the fractional constitutive equation, and the GFE network Zener model is efficient in the modeling of the viscoelastic behavior.

1. Introduction

The stress relaxation, creep, hysteresis and energy dissipation processes of viscoelastic materials such as plastics, resins, rubbers, muscle tissue, and polymers have aroused widespread interests, much research has been done in this field [1–3].

In recent years, fractional calculus has encountered much success in the description of complex dynamics [4–11]. It has been proved that the classical viscoelastic models can be generalized to fractional models to better describe the viscoelastic behavior of materials. By comparison with classical models, the fractional calculus provides an appropriate description of the system behavior using fewer parameters, and the fractional operators with infinite memory of the system can model its actual behavior. For instance, a four-parameter Maxwell model with fractional derivatives is used to determine the relaxation and retardation functions of viscoelastic materials [12]. The fractional Kelvin–Voigt constitutive law is applied to model the damping properties [13]. Besides, the viscoelastic property of Crucian carp is also investigated using a fractional Zener model to fit the relaxation force and the experimental results [14].

The models of fractional elements (FEs) proposed by Schiessel and his co-workers [15–17] are able to depict the piecewise self-similar properties of viscoelastic materials. They symbolize the relation between stress and strain by a triangle or a

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ladder using three parameters (β, τ, E), that is

$$\sigma(t) = E\tau^{\beta}\ {}_0D_t^{\beta}\varepsilon(t), \tag{1}$$

where $\sigma(t)$, $\varepsilon(t)$ are stress and strain, respectively. E is a constant parameter of the FE, τ is the relaxation time and ${}_0D_t^{\beta}$ is Riemann–Liouville (R–L) differential operator [18] with fractional order β, $0 < \beta < 1$, defined as

$${}_0^{RL}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \quad (n-1 \le \alpha < n). \tag{2}$$

The pure elastic and pure viscous situations are in accordance with the cases that $\beta = 0$ and $\beta = 1$. In view of the limitations of FEs, Xu et al. [19] propose a new generalized fractional element (GFE) networks which compose of only one kind of FE. The GFE networks can not only extend the model solutions to the generalized function space but also eliminate the restrictions on the parameters.

An important problem, connected with the fractional rheological models, is the identification of the model parameters from experimental data. Plenty of methods have been developed for classical inverse problems [20,21], while research on that of fractional models is quite little. Nowadays, the Levenberg–Marquardt method is shown to be efficient to estimate the relaxation time and the order of fractional derivative in fractional single-phase-lag heat equation by Ghazizadeh et al. [22]. Work by Ribeiro [23] shows that the parameters in the generalized Maxwell model can be estimated through numerical optimization techniques.

In this paper, attention has been paid to the GFE network Zener model, a Bayesian method to investigate the inverse problem of parameter estimation for the fractional constitutive equation is proposed. In Section 2, a brief review of the model and its analytical solution is given. The theory of the Bayesian method is described in Section 3. In order to prove the effectiveness of the method and the reliability of the estimated results, three examples are performed in Section 4. Finally in Section 5, some conclusions are drawn based on the numerical results.

2. GFE network Zener model and its analytical solution

Fig. 1 shows the fractional Zener model for viscoelastic materials based on the GFE networks. As is well known, the stress is the same for elements in series and the strain is the same for elements in parallel. Without loss of generality, the parameters are assumed to be $0 \le \beta \le \alpha \le 1$ and, of course, $0 \le \lambda \le 1$. Then we can get the stress–strain constitutive equation [19].

$$\sigma(t) + \tau^{\alpha-\beta}\ {}_0D_t^{\alpha-\beta}\sigma(t) = E_0\tau^{\alpha}\ {}_0D_t^{\alpha}\varepsilon(t) + E\tau^{\lambda}\ {}_0D_t^{\lambda}\varepsilon(t) + E\tau^{\lambda+\alpha-\beta}\ {}_0D_t^{\lambda+\alpha-\beta}\varepsilon(t), \tag{3}$$

where

$$\tau = (E_1\tau_1^{\alpha}/E_2\tau_2^{\beta})^{1/(\alpha-\beta)}, \qquad E_0 = E_1(\tau_1/\tau)^{\alpha}, \qquad E = E_3(\tau_3/\tau)^{\lambda}. \tag{4}$$

Assuming that $\varepsilon(t) = H(t)$, where $H(t)$ is Heaviside unit step function. As a Cauchy problem of linear autonomous system with fractional order, the analytical solution to the model can be achieved by taking the Laplace transform to Eq. (3) and then using the discrete method to perform the inverse Laplace transform [19]. The solution of Eq. (3) is given as

$$\sigma(t) = E_0\left(\frac{t}{\tau}\right)^{-\beta} E_{\alpha-\beta,1-\beta}\left(-\left(\frac{t}{\tau}\right)^{\alpha-\beta}\right) + E\frac{\left(\frac{t}{\tau}\right)^{-\lambda}}{\Gamma(1-\lambda)}, \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function and $E_{v,u}(z)$ is the generalized Mittag-Leffler (M-L) function [18] defined by

$$E_{v,u}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(nv+u)}, \quad (z \in C, v > 0, u > 0). \tag{6}$$

In a similar way, as to the other case of $0 \le \alpha \le \beta \le 1$, the solution to the corresponding constitutive equation can also be achieved.

Particularly, in Eq. (5), let $E_3 = 0$, then

$$\sigma(t) = E_0\left(\frac{t}{\tau}\right)^{-\beta} E_{\alpha-\beta,1-\beta}\left(-\left(\frac{t}{\tau}\right)^{\alpha-\beta}\right), \tag{7}$$

this is just the analytical solution for the fractional Maxwell model [15].

In this paper, we mainly consider the fractional stress–strain constitutive equation of the GFE network Zener model as shown in Eq. (3) under the conditions $0 \le \beta \le \alpha \le 1$ and $0 \le \lambda \le 1$ in particular.

3. The Bayesian method for parameter estimation

The research on parameter estimation for fractional rheological models is of great significance. However, there is a few specific inverse methods in this field. As is well known, the Bayesian statistical method has many advantages over other