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# Entropy maximization and instability in uniformly magnetized plasmas

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## HIGHLIGHTS

- A way to overcome the Child–Langmuir law is studied.
- The magnetic mirror effect mitigates the space-charge effect.
- The value of the achievable current depends on the magnetic field configuration.

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ABSTRACT

A regime in which a uniformly magnetized plasma does not maximize the entropy and possibly becomes unstable to a spatial perturbation in the magnetic field is explored. The physical implication is considered in the context of current generation, magnetic field reconnection, and the dynamo effect.



## 1. Introduction

A uniformly magnetized plasma is generally perceived to be dynamically stable: it is not vulnerable to a localized spatial perturbation in the magnetic field [1]. However, this may not always be the case, and if it is indeed true, our premise on a uniform magnetic field would have to be re-examined carefully.

The concept of entropy has been applied to systems of magnetized plasmas [2–4]. The approach to an equilibrium state of a dynamical system is often accounted for by maximizing the system entropy, under a given volume and the total energy [5–7]. The equilibrium configuration of a magnetized one-component electron plasma can be determined by this principle; the electron distribution function can be obtained by maximizing the Shannon entropy [8] in the framework of the variational principle, under the constraints of the system-wide conserved quantities including the gross number of the electrons, the volume, the energy, and the magnetic moment. Consider two different plasmas, one subject to a uniform magnetic field $\mathbf{B}_1 = B_0\hat{z}$ and the other subject to a spatially varying field $\mathbf{B}_2 = B_0(1+\beta\cos(kx))\hat{z}$, where $\beta$ is a small number. If the entropy of the latter system exceeds that of the former, it would attest that the plasma with a uniform magnetic field could transition to a spatially varying state. The goal of this paper is to demonstrate that this is indeed possible in certain regimes.

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## 2. Theoretical analysis

We consider a plasma in which the collision frequency is slower than the other time scales such as the gyro-frequency and the plasma frequency. Then the Shannon entropy remains unchanged under the collisionless Vlasov equation, a usual tool to study the collective electron dynamics. On the other hand, it is also believed that the effect of coarse graining or phase mixing would increase the entropy [6,7]. In the following, we employ the phase-mixing argument and assume that the entropy of the electrons increases by evolving in a way different from what is prescribed by the collisionless Vlasov equation.

Let us denote the plasma with the uniform (spatially varying) magnetic field by the plasma **U** (plasma **V**) and the corresponding entropy by $S_U$ ($S_V$). We assume that both plasmas have identical system-wide quantities, including the total energy, magnetic moment, total number of the electrons, and volume. The total energy of plasma **U** subject to a uniform magnetic field $B_0\hat{z}$ is

$$E_U = \int \frac{B_0^2}{8\pi} d^3\mathbf{x} + \sum_i \frac{1}{2} m_e v_i^2, \tag{1}$$

where $m_e$ is the electron mass and the summation is done over all the constituting electrons, and the interaction between the electron magnetic moment and the magnetic field is ignored. The total magnetic moment of the system is

$$M_U = \sum_i \frac{m_e c}{B_0} v_i^2, \tag{2}$$

and the Shannon entropy is

$$S_U = -\sum_k p_q \log(p_q), \tag{3}$$

where $c$ is the speed of light, $p_q$ is the probability for the plasma to be in the state $q$, and the summation is done over all the possible states. From the variational principle, i.e., $\delta S + l_a \delta E_U + l_b \delta M_U = 0$, where $l_a$ and $l_b$ are the Lagrange multipliers, we obtain for each $q$

$$\delta p_q \left[ -\log(p_q) + l_a E_U + l_b M_U \right] = 0, \tag{4}$$

which leads to an anisotropic Maxwellian distribution

$$f_U(\mathbf{v}) = \frac{1}{\sqrt{2\pi}^3} \frac{n_0}{v_{U\perp}^2 v_{U\parallel}} \exp\left( -\frac{v_x^2 + v_y^2}{2v_{U\perp}^2} - \frac{v_z^2}{2v_{U\parallel}^2} \right), \tag{5}$$

where $v_{U\perp}$ ($v_{U\parallel}$) is the perpendicular (parallel; $z$-direction) thermal velocity and $n_0$ is the electron density. The temperature in each direction is determined by the constraint on the total energy and the total magnetic moment.

Let us now consider the second plasma **V**, subject to a spatially varying magnetic field $B_0(1 + \beta\cos(kx))\hat{z}$, where $\beta \ll 1$. Following the same steps, the electron distribution is obtained to be

$$f_V(\mathbf{v}, x) = \frac{1}{\sqrt{2\pi}^3} \frac{n_V}{v_{V\perp}^2 v_{V\parallel}} \exp\left( -\frac{v_x^2 + v_y^2}{2v_{V\perp}(x)^2} - \frac{v_z^2}{2v_{V\parallel}^2} \right), \tag{6}$$

where $v_{V\perp}$ ($v_{V\parallel}$) is the perpendicular (parallel) thermal velocity

$$v_{V\perp}(x)^2 = \frac{v_{V\perp}^2}{1 + \frac{b\beta\cos(kx)}{1+\beta\cos(kx)}}, \tag{7}$$

where $b = \gamma - 1$ with $\gamma = v_{U\perp}^2 / v_{U\parallel}^2$. The three unknowns, $v_{V\perp}$, $v_{V\parallel}$, and $n_V$, can be determined as a function of $v_{U\perp}$, $v_{U\parallel}$, $n_0$ and $\beta$, given the constraints on the total number of electrons, the energy, and the magnetic moment. Conservation of the number of electrons $\int f_V d^3\mathbf{v} = \int f_U d^3\mathbf{v}$ leads to

$$n_V = \frac{n_0}{1 + b(b+1)\bar{\beta}^2}, \tag{8}$$

where $\bar{\beta}^2 = \langle \beta^2 \cos^2(kx) \rangle = \beta^2/2$, and the constraint on the total magnetic moment

$$\int f_V \frac{v_x^2 + v_y^2}{B_0(1 + \beta\cos(kx))} = \int f_U \frac{v_x^2 + v_y^2}{B_0} \tag{9}$$