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Dynamics of the Langevin model subjected to colored noise: Functional-integral method

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Abstract

We have discussed the dynamics of Langevin model subjected to colored noise, by using the functional-integral method (FIM) combined with equations of motion for mean and variance of the state variable. Two sets of colored noise have been investigated: (a) one additive and one multiplicative colored noise, and (b) one additive and two multiplicative colored noise. The case (b) is examined with relevance to a recent controversy on the stationary subthreshold voltage distribution of an integrate-and-fire model including stochastic excitatory and inhibitory synapses and a noisy input. We have studied the stationary probability distribution and dynamical responses to time-dependent (pulse and sinusoidal) inputs of the linear Langevin model. Model calculations have shown that results of the FIM are in good agreement with those of direct simulations (DSs). A comparison is made among various approximate analytic solutions such as the universal colored noise approximation (UCNA). It has been pointed out that dynamical responses to pulse and sinusoidal inputs calculated by the UCNA are rather different from those of DS and the FIM, although they yield the same stationary distribution.

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Keywords: Langevin model; Colored noise; Functional-integral method

1. Introduction

Nonlinear stochastic dynamics of physical, chemical, biological and economical systems has been extensively studied (for a recent review, see Ref. [1]). In most theoretical studies, Gaussian white noise is employed as random driving force because of its mathematical simplicity. The white-noise approximation is appropriate to systems in which the time scale characterizing the relaxation of the noise is much shorter than the characteristic time scale of the system. There has been a growing interest in the theoretical study of nonlinear dynamical systems subjected to colored noise with the finite correlation time (for a review on colored noise, see Ref. [2] and the related references therein). It has been realized that colored noise gives rise to new intriguing effects such as the reentrant phenomenon in a noise-induced transition [3] and a resonant activation in bistable systems [4].

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The original model for a system driven by colored noise is expressed by non-Markovian stochastic differential equation. This problem may be transformed to a Markovian one, by extending the number of relevant variables and including an additional differential equation describing the Orstein–Uhlenbeck (OU) process. It is difficult to analytically solve the Langevin model subjected to *colored* noise. For its analytical study, two approaches have been adopted: (1) to construct the multi-dimensional Fokker–Planck equation (FPE) for the multivariate probability distribution, and (2) to derive the effective one-dimensional FPE equation. The presence of multi-variables in the approach (1) makes a calculation of even the stationary distribution much difficult. In a recent study on the Langevin model subjected to additive (non-Gaussian) colored noise [5], we employed approach (1), analyzing the multivariate FPE with the use of the second-order moment method. A typical example of approach (2) is the universal colored noise approximation (UCNA) [6], which interpolates between the limits of zero and infinite relaxation times, and which has been widely adopted for a study of colored noise [2]. Another example of the approach (2) is the path-integral and functional-integral methods [7–12] obtaining the effective FPE, with which stationary properties such as the non-Gaussian stationary distribution have been studied [2].

Theoretical study on the Langevin model driven by colored noise has been mostly made for its stationary properties such as the stationary probability distribution and the phase diagram of noise-induced transition [2]. As far as we are aware of, little theoretical study has been reported on dynamical properties such as the response to time-dependent inputs. Refs. [13,14] have discussed the filtering effect, in which the high-frequency response of the system is shown to be improved by colored noise. The purpose of the present paper is to extend the functional-integral method (FIM) such that we may discuss the dynamical properties of the Langevin model subjected to colored noise. We consider, in this paper, two sets of colored noise: (a) one additive and one multiplicative colored noise, and (b) one additive and two multiplicative colored noise. The case (b) is included to clarify, to some extent, a recent controversy on the subthreshold voltage distribution of a leaky integrate-and-fire model including conductance-based stochastic excitatory and inhibitory synapses as well as noisy inputs [15–18].

The paper is organized as follows. The FIM is applied to the above-mentioned cases (a) and (b) in Sections 2 and 3, respectively, where the stationary distribution and the response to time-dependent inputs are studied. In Section 4, we will discuss the recent controversy on the subthreshold voltage distribution of a leaky integrate-and-fire model [15–18]. A comparison is made among the results of some approximate analytical theories such as the UCNA [2,6]. The final Section 5 is devoted to conclusion.

2. Langevin model subjected to one additive and one multiplicative colored noise

2.1. Effective Langevin equation

We have considered the Langevin model subjected to additive and multiplicative colored noise given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) + \eta_0(t) + G(x)\eta_1(t), \tag{1}$$

with

$$\frac{\mathrm{d}\eta_m(t)}{\mathrm{d}t} = -\frac{\eta_m}{\tau_m} + \frac{\sqrt{2D_m}}{\tau_m}\,\xi_m(t), \quad (m = 0 \text{ and } 1) \tag{2}$$

where $F(x)$ and $G(x)$ denote arbitrary functions of x: $\eta_0(t)$ and $\eta_1(t)$ stand for additive and multiplicative noise, respectively: τ_m and D_m express the relaxation times and the strengths of colored noise for additive ($m = 0$) and multiplicative noise ($m = 1$): $\eta_m(t)$ express independent zero-mean Gaussian white noise with correlations given by

$$\langle \xi_m(t)\xi_n(t')\rangle = \delta_{mn}\delta(t - t'). \tag{3}$$

The distribution and correlation of η_m are given by

$$p(\eta_m) \propto \exp\left(-\frac{\tau_m}{2D_m}\eta_m^2\right), \tag{4}$$

$$c_{mn}(t, t') = \langle \eta_m(t)\eta_n(t')\rangle = \delta_{mn}\left(\frac{D_m}{\tau_m}\right)\exp\left(-\frac{\mid t - t' \mid}{\tau_m}\right). \tag{5}$$