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Physica A 387 (2008) 2750–2760
PHYSICA A
www.elsevier.com/locate/physa

A continuous variant for Grünwald–Letnikov fractional derivatives

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Received 10 September 2007; received in revised form 11 January 2008
Available online 26 January 2008

Abstract

The names of Grünwald and Letnikov are associated with discrete convolutions of mesh h, multiplied by $h^{-\alpha}$. When h tends to zero, the result tends to a Marchaud's derivative (of the order of α) of the function to which the convolution is applied. The weights w_k^α of such discrete convolutions form well-defined sequences, proportional to $k^{-\alpha-1}$ near infinity, and all moments of integer order $r < \alpha$ are equal to zero, provided α is not an integer. We present a continuous variant of Grünwald–Letnikov formulas, with integrals instead of series. It involves a convolution kernel which mimics the above-mentioned features of Grünwald–Letnikov weights. A first application consists in computing the flux of particles spreading according to random walks with heavy-tailed jump distributions, possibly involving boundary conditions.

PACS: 05.60.-k; 46.65.+g; 05.40.Fb; 02.60.Nm

Keywords: Transport processes; Random media; Random walks; Integro-differential equations

1. Introduction

A variety of mathematical objects denoted as derivatives, and qualified by the word "fractional", were built and studied in view of heterogeneous motivations. The subsequent confusion, which contrasts with what occurred for derivatives of integer order, led to fractional calculus becoming unpopular. Yet, the many faces of the notion are present in the various applications, ranging among quantum mechanics [1], medicine [2], geophysics [3,4] and many other fields. Celebrated examples of fractional derivatives solved integral problems, such as e.g. Abel's tautochrone, and several reference textbooks [5–8] have put forward fractional derivatives that invert integral mappings. The derivatives of Riemann–Liouville, Marchaud, Weyl or Caputo are (left or right) inverses to fractional integrals involving integration over various types of intervals, in different function spaces, but they also were given a meaning in various fields of physics. Here we address the set of mappings inverting (at the left) fractional integrals. For the

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2008.01.090

moment, we restrict to the one-dimensional case, and focus on those operators that involve integrations over semi-infinite intervals, whose one limit is the point where we intend to compute them. They commute with translations, the inverses too, which property they share with regular derivatives. Left inverses to such mappings can be given by explicit formulas, such as that of Riemann–Liouville, or Marchaud which is more general. They coincide with Grünwald–Letnikov formulas, which are at the basis of frequently used numerical approximations to fractional derivatives.

We show that those mappings also coincide with the limit, when h tends to zero, of

$$h^{-\alpha}\int_0^{+\infty} f(x \pm hy)F(y)\mathrm{d}y, \tag{1}$$

which (1) generalizes the Grünwald–Letnikov approach, using continuous instead of discrete convolution (with integrals instead of series). In other words, the limit of (1) is the inverse of left- or right-sided fractional integrals of the order of α, associated with integration over $]x, +\infty[$ and $]-\infty, x[$. Hence, for all positive α, it represents Marchaud's derivatives (of the order of α) of function f. It yields quite a number of approximations to fractional derivatives, since kernel F only has to match oscillation conditions and to behave asymptotically as Grünwald–Letnikov weights. It generalizes to all positive orders a result, previously obtained for α between 0 and 1 [9], and presented in a slightly different form. For those values of α, the integral on the right-hand side of (1) represents an essential step in computing fluxes of particles performing random walks [10–13], allowing for heavy tails connected with the value of α. In this context, parameter h in (1) represents a length scale for trajectories composed up of successive independent random jumps, and function f stands for the density of particles. Kernel F is tightly connected with the probability for a given jump to have an amplitude of more than a given value, and prefactor $h^{-\alpha}$ corresponds to a scale, typical of the pausing times between successive jumps, which we suppose here to be equal to τ. Assuming that they are realizations of a random variable of mean τ as in Ref. [9] would yield a similar issue. The prefactor in front of the integrals comes up naturally when we count particles crossing a given location per unit of time, provided we assume a scaling law of the form of $h^\alpha/\tau = K$ [14]. Here K generalizes the classical diffusivity of the Brownian case. Passing to the limit in (1) is equivalent to taking the macroscopic point of view, at which characteristic scales of particle motions are small. Formula (1) helps showing that, when the spreading of matter can be modeled by random walks as above, the flux is a linear combination of fractional derivatives, possibly involving also local derivatives, in fact equal to zero where the density of particles is smooth. Nevertheless, the latter may be visible at singularities, such as sources, sinks or boundaries.

Before detailing the mathematical result, we recall basic points, useful for our purpose and concerning the inverses of those fractional integrals, commuting with translations. Then, we state conditions ensuring that the limit of (1) when h tends to zero is a fractional derivative. From this we deduce a fractional variant of Fick's law for heavy-tailed random walks.

2. Riemann–Liouville and Marchaud fractional integrals and derivatives

In analogy with the many paths, connecting a given discrete set of points, quite a number of mappings interpolate between derivatives of integer orders. Among them, Riemann–Liouville and Marchaud derivatives are intimately bound to fractional integrals over semi-infinite intervals, for which they play the role of (left) inverse mappings.

2.1. Riemann–Liouville and Marchaud derivatives

For α being a positive real number, the left- and right-sided fractional integrals of the order of α of function f, associated with intervals $\mathcal{I}_+ =]a, x]$ and $\mathcal{I}_- = [x, b[$ are [5–8]

$$I^\alpha_\pm f(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathcal{I}_\pm}(x-y)^{\alpha-1}f(y)\mathrm{d}y, \tag{2}$$

which interpolates between integrals of integer orders. Properties of fractional integrals as (2) are detailed in Refs. [5–8], and here we restrict to $a = -\infty$ and $b = +\infty$. In this context, Riemann–Liouville left- and right-sided derivatives