Physica A 422 (2015) 187–192

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A definition of the coupled-product for multivariate coupled-exponentials

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HIGHLIGHTS

- A generalized product is defined to factor multivariate coupled exponentials.
- The coupling is defined independent of the power and dimensions of the variable.
- The output dimension of the generalized product is the sum of the input dimensions.

ARTICLE INFO

Article history:
Received 15 September 2014
Received in revised form 15 December 2014
Available online 29 December 2014

Keywords:
Nonextensive statistical mechanics
Non-additive entropy
Multivariate distributions
Nonlinear coupling
Complex systems

ABSTRACT

The coupled-product and coupled-exponential of the generalized calculus of nonextensive statistical mechanics are defined for multivariate functions. The nonlinear statistical coupling is indexed such that $\kappa_d = \kappa/1 + d\kappa$, where d is the dimension of the argument of the multivariate coupled-exponential. The coupled-Gaussian distribution is defined such that the argument of the coupled-exponential depends on the coupled-moments but not the coupling parameter. The multivariate version of the coupled-product is defined such that the output dimensions are the sum of the input dimensions. This enables construction of the multivariate coupled-Gaussian from univariate coupled-Gaussians. The resulting construction forms a model of coupling between distributions, generalizing the product of independent Gaussians.

1. Introduction

A definition for a generalization of the exponential family based on the degree of nonlinear statistical coupling is provided. The purpose of the definition is to facilitate development of a generalized calculus which will enable analytic reasoning for nonlinear systems, while retaining as much of the structure of the exponential family and its connection to linear systems. The definition builds upon the generalized calculus of nonextensive entropy [1,2], but incorporates several new features. First, the proposed generalized calculus is defined using the nonlinear statistical coupling $\kappa = 1 - q$ [3], a translation of the Tsallis entropy index [4]. This translation has also been utilized by for example Refs. [1,5–7]. Second, the coupling terms for the arguments and the exponent have a non-equal functional relationship [8]. Third, the two coupling terms are defined such that the argument of the coupled-exponential and coupled-Gaussian distribution only depend on the generalized parameters of the distribution. Fourth, the dependence on the dimensions of the argument is separated from the coupling. The purpose of these modifications is to isolate the physical effects of the nonlinear coupling and to define the coupled-product consistent with the change in dimensions of a multivariate coupled-exponential [8] from the operands to the resultant.

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http://dx.doi.org/10.1016/j.physa.2014.12.023

2. Definitions of the coupled-exponential and related algebra

The concept of nonlinear statistical coupling is based on two mathematical features of the generalized calculus for nonextensive statistical mechanics. Firstly, the coupled-sum $x \oplus_\kappa y = x + y + \kappa xy$ modifies addition to include a nonlinear term weighted by the coupling κ. Secondly, the escort or coupled-probability $P_i^{(\kappa)} \equiv p_i^{1-\kappa} / \sum_{j=1}^n p_j^{1-\kappa}$ can be expressed as measuring the probability of a coupling of the states of a system. This is demonstrated by multiplying the numerator and denominator by $\prod_{k=1}^n p_k^\kappa$:

$$P_i^{(\kappa)} = \frac{p_i^{1-\kappa}}{\sum\limits_{j=1}^n p_j^{1-\kappa}} = \frac{p_i \prod\limits_{\substack{k=1 \\ k\neq i}}^n p_k^\kappa}{\sum\limits_{j=1}^n \left(p_j \prod\limits_{\substack{k=1 \\ k\neq j}}^n p_k^\kappa \right)}. \tag{2.1}$$

The expression on the right represents the normalized probability of the independent occurrence of event i and κ occurrences of the all the other events.

The definition for the coupled-exponential $\exp_\kappa(x) \equiv (1+\kappa x)_+^{1/\kappa}$; $(a)_+ \equiv \max(0, a)$ when applied to multivariate distributions has the following difficulties. The argument of the coupled-Gaussian, $x = \frac{-y^2}{(2+\kappa)\sigma_\kappa^2}$, depends on both the coupling itself and the coupled-variance, $\sigma_\kappa^2 = \frac{\int_{-\infty}^{\infty} x^2 f^{1-\kappa}(x)\mathrm{d}x}{\int_{-\infty}^{\infty} f^{1-\kappa}(x)\mathrm{d}x}$, where $\int_{-\infty}^{\infty} f^{1-\kappa}(x)\,\mathrm{d}x / \int_{-\infty}^{\infty} f^{1-\kappa}(x)\,\mathrm{d}x$ is the continuous form of the coupled-probability. A definition which allowed the argument to be independent of the coupling would better express the generalization. Secondly, for multiple dimensions the coupling is a function of the number of dimensions. Because of the dependence on the dimensions, the coupled-product $x \otimes_\kappa y \equiv (x^\kappa + y^\kappa - 1)^{1/\kappa}$ cannot be used to factor a multivariate distribution into the appropriate marginal distributions, since the marginals have different dimensions and hence different coupling values. These issues are addressed by defining the coupled-exponential such that the coupling term multiplying the argument and the coupling term forming the power are not necessarily equal.

Definition 1 (*Coupled-Exponential*)**.** Given a variable x, the nonlinear statistical coupling $\kappa \in \mathbb{R}$ and an additional coefficient $a \in \mathbb{R}$ the *coupled-exponential* is defined as

$$\mathrm{e}_{\kappa_a}^x \equiv \exp_{\kappa_a}(x) \equiv \exp_\kappa(x; a) \equiv (1+\kappa x)_+^{\frac{1}{\kappa_a}} \tag{2.2}$$

$$\kappa_a \equiv \left(\frac{1}{\kappa} + a\right)^{-1} \tag{2.3}$$

where $(y)_+ \equiv \max(0, y)\ \forall y \in R$.

For finite a, $\lim_{\kappa\to 0} \mathrm{e}_{\kappa_a}^x = \mathrm{e}^x$ and for $a = 0$, $\mathrm{e}_{\kappa_0}^x = \mathrm{e}_\kappa^x = (1+\kappa x)_+^{1/\kappa}$ which is the original definition of the coupled-exponential. This more general definition will facilitate development of a generalized algebra for coupled random variables which are multivariate. The inverse of this function is the coupled-logarithm.

Definition 2 (*Coupled-Logarithm*)**.** Given a variable $x > 0$ and nonlinear statistical coupling $\kappa \in \mathbb{R}$, the *coupled-logarithm* is defined as

$$\ln_{\kappa_a} x \equiv \ln_\kappa(x; a) \equiv \frac{1}{\kappa}\left(x^{\kappa_a} - 1\right). \tag{2.4}$$

Anticipating the connection with dimensions, the coupled-product is defined such that the output coupling parameter is indexed by the sum of the input indices. The definition is expressed using functions as the input, which later will be explicitly multidimensional distributions, but can be any scalar.

Definition 3 (*Multivariate Coupled-Product*)**.** Given two functions $f_{1,2}$ and a coupling κ their coupled-product $\otimes_{(\kappa_{a_1}, \kappa_{a_2})}$ is

$$f_1 \otimes_{(\kappa_{a_1}, \kappa_{a_2})} f_2 \equiv \left(f_1^{\kappa_{a_1}} + f_2^{\kappa_{a_2}} - 1\right)_+^{1/\kappa_{a_1+a_2}}. \tag{2.5}$$

The extension of the coupled-product to multiple functions is given by

$$\prod_{i=1}^n \otimes_{\kappa_{a_i}} f_i \equiv \left(\sum_{i=1}^n f^{\kappa_{a_i}} - (n-1)\right)^{1/\kappa_{\sum a_i}}. \tag{2.6}$$