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Volatility estimators and the inverse range process in a random volatility random walk and Wiener processes

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Abstract

The purpose of this paper is to study the mean, the variance, the probability distribution and the hazard rate of the inverse range process of an a-priori unknown volatility random walk. Motivation for this process arises when it is necessary to obtain statistics that pertain to a process volatility in addition to the usual variance statistics. As a result, range process statistics are indicated as an additional source of information in the study of processes' volatility. Examples and applications are considered.

Keywords: Volatility; Range process; Risk

1. Introduction

Empirical evidence has shown that financial time series are not always "well behaved". They may have an unpredictable variance, underscoring departures from the "random walk hypothesis". These effects have been recognized and have been the subject of considerable research under the heading of ARCH- and GARCH-related models [5,2,3] focusing on the estimation of an underlying process variance. A related effort based on samples range of a constant volatility random walk was pointed out by Parkinson [10] however. Explicitly, using Feller's result (1957) Parkinson's volatility estimator is given by

$$\hat{\sigma}^2(p) = \frac{1}{(4\ln 2)Tn}\sum_{i=1}^{n} R_i$$

where $R_i = \max_{i\in I_i} X(t) - \min_{i\in I_i} X(t)$ is the ith sample range, n is the number of intervals I_i, $i = 1, 2 \ldots, n$ over which the range is estimated, T is the length of the interval while $X(t)$ is assumed to be a normal process with volatility σ. An adjustment to this estimate, based on the transformation of Parkinson's samples is suggested by Kunitomo [8], providing thereby an estimate which is equivalent to the estimates based on samples variances. Range-based estimates

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in econometrics have been studied and applied further by Martens and van Dijk [9] as well as Alizadeh et al. [1] and Brandt and Diebold [4] who claim that the daily range in financial data is a more robust estimator against the effects of microstructure noise than the realized variance. By contrast, in the study of data with long memory, the Hurst index based on a range to standard deviation statistic provides the means to estimate a latent long-run memory in long-time series (for example, Ref. [11]).

The purpose of this paper is to provide volatility estimators in stationary random volatility models using instead an inverse range process statistic which will be defined here. The intent of the paper is to indicate the importance of such a (range) process to estimate directly the functional parameters of an unknown underlying volatility and provide statistical estimates for this volatility. Unlike the ARCH-GARCH approach which is used to counter the effects of heteroscedasticity in linear regressions (where volatility is random), and uses specific models of volatility to negate their statistical effects, our approach is based on the information that a range process can provide to estimate the underlying process volatility. Our approach therefore is not a substitute to ARCH and GARCH estimation techniques but complements it when the volatility is unknown or is indirectly observable. Of course, if the process can be directly observed and measured, such an estimate can be used in addition to the standard maximum likelihood techniques applied in estimating directly the volatility. For example, random (stationary) volatility (and variance) distributions such as the exponential, the lognormal, Cauchy (fat tail) and other distributions can be used as hypotheses regarding an underlying volatility process and the parameters estimated by an observation of the range.

Unlike papers by Feller [6], Vallois [12], Vallois and Tapiero [13–15], we assume that the random walk process volatility is characterized by a known probability distribution and calculate its essential statistics (although approximations can be reached when such a distribution is not specified). Both discrete and continuous time observations of a given range may be used in our calculations. As a result, both random walk and random Wiener processes are considered here.

Problems related to volatility (variance) estimation are particularly important in finance. For example, an investor—a buyer of options, may have only a probabilistic assessment of the underlying stock volatility, which is essential for option's pricing. Random volatility tests are also an indicator of markets' incompleteness and therefore important to detect arbitrage opportunities. Similarly, volatility estimates are used in control charts to determine control limits (often based as well on sample ranges which assume that samples are iid). In such cases, process range statistics might be used to test the validity of a given set of control limits. We begin by developing estimators for a random volatility random walk and obtain an explicit expression to estimate the parameters of the volatility distribution. Subsequently, a random volatility Wiener process is considered and specific cases (volatility distributions and estimators) are resolved.

2. A random volatility random walk

Consider the symmetrical random walk, initiated at zero:

$$X_0 = 0, \qquad X_n = \sum_{i=1}^{n} \varepsilon_i, \quad n \geq 1 \tag{1}$$

where $(\varepsilon_i)_{i \geq n}$ is a series of independently distributed random variables with $P(\varepsilon_i = \pm 1) = 1/2$. We define as well the random (volatility) parameter, $\alpha > 0$ independent of the random walk $(X_n)_{n \geq 0}$ and consider the following process:

$$X_n^{(\alpha)} = \alpha X_n; \quad n \geq 0. \tag{2}$$

Such a process implies that the underlying random walk (a price for example) increases or decreases by increments of size α. When α is unknown, we then have a constant volatility process—albeit its parameters are unknown and therefore presumed to be defined by a probability distribution. When α is a stochastic process, say a Bernoulli process assuming values a with probability p and zero otherwise, the underlying process (2) becomes a trinomial random walk with unknown parameters (a, p) that may be determined in the same manner that α is estimated in this paper. However, such extensions and some of their difficulties are discussed in the appendix and provide an area for further research. In particular, while the trinomial random walk is the sum of independent random variables, the random walk with a Bernoulli mixture results in a mixture distribution where the random increments are dependent. For our current purpose, we rewrite (2) as follows:

$$X_n^{(\alpha)} = X_{n-1}^{(\alpha)} + \alpha \varepsilon_n \quad \text{or} \tag{3}$$