Physica A 392 (2013) 2434–2441

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Distribution of wealth in a network model of the economy

Tao Ma [a], John G. Holden [b], R.A. Serota [a,*]

[a] Department of Physics, University of Cincinnati, Cincinnati, OH 45244-0011, United States

[b] CAP Center for Cognition, Action, and Perception, Department of Psychology, University of Cincinnati, Cincinnati, OH 45244-0376, United States

ARTICLE INFO

Article history:
Received 23 August 2012
Available online 24 January 2013

Keywords:
Network model
Small world network
Wealth distribution
Mean field theory
Effective field theory
Generalized inverse gamma distribution
Pareto
Power-law tail
Fokker–Planck equation

ABSTRACT

We show, analytically and numerically, that wealth distribution in the Bouchaud–Mézard network model of the economy is described by a three-parameter generalized inverse gamma distribution. In the mean-field limit of a network with any two agents linked, it reduces to the inverse gamma distribution.

1. Introduction

Wealth distribution has become a subject of keen interest in econophysics research [1]. Here, we study a network model of the economy proposed by Bouchaud and Mézard (BM) [2]. In the mean field (MF) limit of a completely connected network, where any two agents in the network are linked, the model yields the inverse gamma (IGa) stationary wealth distribution. An important feature of the IGa distribution is the power-law (PL) tail [3]. In the opposite limit of a completely disconnected network, the time-dependent part of wealth distribution is lognormal (LN). Both LN and IGa have long histories in models of wealth distribution. LN is generated by Gibrat's law [4]. IGa, as well as a specific form of GIGa (generalized inverse gamma distribution), was used to analyze wealth distribution in ancient Egypt [5].

Souma et al. did a numerical study of the BM model and proposed that there may be quite an abrupt transition between IGa and LN as a function of the number of connections and the type of connections—regular network or a small-world network [6]. In this paper, we revisit Souma's simulations and compute the p-values of the fitting distributions using the Kolmogorov–Smirnov test [7]. We argue that the time-dependent LN distribution is a transient – albeit possibly slow, depending on the parameters – and concentrate on the stationary solution. We find that for the BM model the latter is a three-parameter GIGa distribution. Theoretically, we develop an effective field theory for the BM model of partially connected networks, including regular network and a random small-world network [6] and obtain the Fokker–Planck equation (FPE) for the probability density function (PDF). Its stationary solution is a GIGa distribution, with IGa distribution as its limit in the MF regime.

* Corresponding author. Tel.: +1 2835135560538; fax: +1 2835135563425.
E-mail addresses: john.holden@uc.edu (J.G. Holden), serota@ucmail.uc.edu (R.A. Serota).

0378-4371/$ – see front matter
doi:10.1016/j.physa.2013.01.045

This paper is organized as follows. In Section 2, we discuss the effective field theory of the BM model, the corresponding stationary FPE and its GIGa solution. In Section 3, we present the results of our numerical simulations. In Section 4, we summarize our findings and outline future directions of our work.

2. Theory

2.1. GIGa from the Bouchaud–Mézard model

The BM model reads [2]:

$$\mathrm{d}W_i \overset{S}{=} \sqrt{2}\sigma W_i \mathrm{d}B_i + \sum_{j(\neq i)} J_{ij} W_j \mathrm{d}t - \sum_{j(\neq i)} J_{ji} W_i \mathrm{d}t, \tag{1}$$

where $\overset{S}{=}$ means that the stochastic differential equation (SDE) is interpreted in the Stratonovich sense [2,6] and $i = 1, 2, \ldots, N$ with $N \gg 1$ the total number of agents, W_i is the wealth of an agent, $\mathrm{d}B_i$ is an independent Wiener process and σ and J_{ij} are constants. Since the BM model may have a wider range of applications – including possibly neural networks – than originally intended, we will study it without applying specific interpretations to W and the model parameters.

The BM model in (1) can be rewritten into an Ito SDE [8]:

$$\mathrm{d}W_i \overset{I}{=} \sqrt{2}\sigma W_i \mathrm{d}B_i + \sigma^2 W_i \mathrm{d}t + \sum_{j(\neq i)} J_{ij} W_j \mathrm{d}t - \sum_{j(\neq i)} J_{ji} W_i \mathrm{d}t \tag{2}$$

in agreement with Souma et al. [6]. Rescaling per $W_i(t) = w_i(t)\mathrm{e}^{\sigma^2 t}$, we obtain

$$\mathrm{d}w_i \overset{I}{=} \sqrt{2}\sigma w_i \mathrm{d}B_i + \sum_{j(\neq i)} J_{ij} w_j \mathrm{d}t - \sum_{j(\neq i)} J_{ji} w_i \mathrm{d}t. \tag{3}$$

It is easily seen that in the large N limit, $\sum_{i=1}^{N} \mathrm{d}w_i = \sqrt{2}\sigma \sum_{i=1}^{N} w_i \mathrm{d}B_i \approx 0$, which implies that the total "wealth" fluctuates around a constant value.

Ultimately, the goal is to determine the PDF $P(w, t)$. Towards this end we notice that there is a discontinuous transition from the interacting case $J_{ij} \neq 0$ to the non-interacting case $J_{ij} = 0$, that is, as soon as the interaction between the agents is turned on, the nature of the distribution function is qualitatively changed. Indeed, as follows from Eq. (7.8) in Ref. [8], $P(w, t)$ does not have a stationary limit for $J_{ij} = 0$ and decreases to zero for any finite w when $t \to +\infty$ (while preserving the total "wealth"):

$$P(w, t) = \frac{1}{2\sqrt{\pi t}\sigma w} \exp\left[-\frac{1}{2}\left(\frac{\log w + \sigma^2 t}{\sqrt{2t}\sigma}\right)^2\right]. \tag{4}$$

Conversely, in the $J_{ij} \neq 0$, a stationary solution $P(w) \equiv P(w, \infty)$ exists and in what follows we concentrate on its analytical derivation while leaving dynamics to numerical investigation.

The MF limit of a completely connected network was studied in Ref. [2]. Substituting $J_{ij} = J/N$ in (3) and extending summation on j to each member of the network, we obtain

$$\mathrm{d}w_i \overset{I}{=} \sqrt{2}\sigma w_i \mathrm{d}B_i + J(\overline{w} - w_i)\mathrm{d}t, \tag{5}$$

where $\overline{w} = N^{-1}\sum_{i=1}^{N} w_i$ is the average of w_i. The corresponding FPE is given by

$$\frac{\partial P}{\partial t} = \frac{\partial\{[J(w - \overline{w}) + \sigma^2 w]P\}}{\partial w} + \sigma^2 \frac{\partial}{\partial w}\left[w \frac{\partial(wP)}{\partial w}\right]. \tag{6}$$

Rescaling via $w \to w/\overline{w}$ so that $\overline{w} = 1$, we find the normalized stationary IGa solution [2]

$$P(w) = \frac{\left(\frac{\sigma^2}{J}\right)^{-\frac{J+\sigma^2}{\sigma^2}}}{\Gamma\left(\frac{J+\sigma^2}{\sigma^2}\right)} \mathrm{e}^{-\frac{J}{\sigma^2} w^{-1}} w^{-2-\frac{J}{\sigma^2}} \tag{7}$$

with a PL tail $\propto w^{-(2+J/\sigma^2)}$.

For a partially connected network, where each agent is connected with $1 \leq n = zN \leq (N-1)$ other agents $(0 < z < 1)$, we substitute $J_{ij} = J/n$ in (3) and notice that

$$\sum_{\text{interacting agents: } j(\neq i)} J_{ij}(w_j - w_i) = J(\overline{w}^{(n)} - w_i), \tag{8}$$