Physica A 414 (2014) 137–142

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Lack of self-averaging in random systems—Liability or asset?

Avishay Efrat [a,*], Moshe Schwartz [b]

[a] Department of Exact Sciences, Afeka Tel-Aviv Academic College of Engineering, Tel-Aviv 6910717, Israel
[b] School of Physics and Astronomy, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel

HIGHLIGHTS

- We find that certain bounded quantities are not self-averaging when the correlation length becomes of the order of the size of the system.
- The lack of self-averaging, expressed in terms of properly chosen signal-to-noise ratios, serves to identify phase boundaries.
- By using such signal-to-noise ratios we identify the boundary of the ferromagnetic phase of the random field Ising system.
- Finally we argue that the signal-to-noise behaves in a generic way, independent of the specific physical quantity for which it is obtained.

ARTICLE INFO

Article history:
Received 22 May 2013
Received in revised form 8 April 2014
Available online 7 July 2014

Keywords:
Self-averaging
Signal-to-noise
Correlations
Phase-transition
Ising
Random

ABSTRACT

The study of quenched random systems is facilitated by the idea that the ensemble averages describe the thermal averages for any specific realization of the couplings, provided that the system is large enough. Careful examination suggests that this idea might have a flaw, when the correlation length becomes of the order of the size of the system. We find that certain bounded quantities are not self-averaging when the correlation length becomes of the order of the size of the system. This suggests that the lack of self-averaging, expressed in terms of properly chosen signal-to-noise ratios, may serve to identify phase boundaries. This is demonstrated by using such signal-to-noise ratios to identify the boundary of the ferromagnetic phase of the random field Ising system and compare the findings with more traditional measures.

Quenched random systems like random field systems, random bond systems, spin glasses, etc., are known to be extremely difficult, because of the necessity to perform quenched averages. In fact, it has to be realized, of course, that this necessity is not of physical origin but rather of mathematical convenience, since a given chunk of matter that is measured experimentally has a given single realization of the disorder. The idea of self-averaging allowing ensemble averaging is based on the fact that we are dealing with very large systems [1]. It is argued that the system can be broken up into subsystems large enough to be considered independent of each other. This basic assumption indicates that phenomena connected with breakdown of self-averaging may appear, as the correlation length will become of the order of the linear size of the system. Indeed, such behavior was first observed by Dayan et al. [2], using a technique first suggested by Berker and Ostlund [3]. The subject of breakdown of self-averaging was treated later in a number of papers devoted just to that phenomenon [4–7]. Obviously, lack of self-averaging results in severe difficulties but the fact that it is connected with the divergence of the correlation length suggests that it can provide, perhaps, an independent measure to distinguish between the disordered and the ordered phases. In this article we show that this is indeed the case.

* Corresponding author. Tel.: +972 507757665; fax: +972 37367812.
E-mail addresses: avishaye@bezeqint.net, avishaye@afeka.ac.il (A. Efrat).

http://dx.doi.org/10.1016/j.physa.2014.06.071
0378-4371/

To be specific, consider the random field Ising model described by the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - \sum_i h_i s_i. \tag{1}$$

The pair $\langle i, j \rangle$ denotes a nearest-neighbor pair on a cubic lattice. The field configuration is assumed to be governed by the distribution,

$$P\{h\} = \prod_i \frac{1}{\sqrt{2\pi h^2}} \exp\left(-\frac{{h_i}^2}{2h^2}\right). \tag{2}$$

We take J and h to have the dimensions of energy.

Consider now some thermal average x and its ensemble average, $x_E \equiv [x]$, where $[\ldots]$ denotes ensemble average. Define next the variance of x within the ensemble,

$$\sigma_x \equiv \{[x^2] - {x_E}^2\}^{\frac{1}{2}}. \tag{3}$$

Next, a parameter γ describing the strength of self-averaging in the system (or rather its weakness) is defined as the signal-to-noise ratio

$$\gamma_x \equiv \frac{|x_E|}{\sigma_x}. \tag{4}$$

The signal-to-noise ratio, γ, as defined above, somewhat resembles the Binder cumulant [8,9], as defined for disordered systems, though it is simpler and more intuitive.

It is clear that for the disordered phase γ is infinite in the infinite system. As our former discussion suggests, that possible breakdown of self-averaging may be traced back to situations where the correlation length, ξ, becomes of the order of the linear size of the system. The usual state of affairs is that the correlation length diverges just at the boundary between the ordered and disordered phases and consequently any breakdown of self-averaging may be observed only in the vicinity of the boundary [4–6]. In the random field case the situation is quite different. The correlation length is of the order of the linear size of the system everywhere in the ordered phase and not just at the boundary of the phase [2].

We describe next our procedure for obtaining numerically the relevant physical quantities, which is based on Casher–Schwartz RSRG [10]. As other real space techniques, it provides simple, one step, recursion relations for translational invariant systems that enable the extraction of the critical exponents. For random systems, the recursion relations, obtained using any renormalization scheme, involve the distribution of couplings or, equivalently, all the parameters defining it (e.g. moments, correlations, etc.). In this approach, the recursion relations are truncated to obtain relations involving only the mean and the variance and keeping the random couplings independent [11–14]. The method suggested by Berker and Ostlund [3] overcomes the most difficult problem arising in the approach described above of ignoring the correlations generated by the renormalization procedure (or projecting them on the variance [11]). A given realization is chosen on a finite system. Renormalization is then used to reduce the size of the system to a size where brute force calculation is possible. (The fact that a lot can be learned from considering specific realizations is also stressed in a paper on the RFIM by Wu and Machta [15].) After the required thermal averages are obtained for a given configuration of the disorder, the ensemble average is obtained by repeating the procedure for many configurations (here 10,000) and averaging. Only thermal averages of functions of the spins surviving the renormalization seem to be obtainable directly. Berker and coworkers [16–18] were using, however, the chain rule, applied to first derivatives, to approximately recover thermodynamic densities of the original system from the renormalized couplings of the reduced system. This enables to obtain averages which involve not only surviving spins. As long as higher derivatives are avoided, this procedure is limited, however, to thermal averages of spin products which already appear in the original Hamiltonian (such as nearest-neighbor pair products). In the following we will calculate averages involving non-surviving spins using our "sites translation method" [19].

We start with a set of $N = L^3$ Ising spins, with $L = 2^n$, situated on a three dimensional cubic lattice with periodic boundary conditions. Here $L = 32 = 2^5$ so that $N = 32^3 = 32,768$. We generate a realization of the distribution (2) and then perform the Casher–Schwartz procedure [10] $n-1$ times. At each step of the renormalization, the lattice remains cubic while its linear size is reduced by a factor of two. The couplings remain nearest-neighbor but become position dependent and each spin is affected by a renormalized position dependent field. Finally, we are left with a system of $2 \times 2 \times 2$ spins on which we perform direct calculations. Consequently we obtain 8 $\langle s_i \rangle$'s, corresponding to the spins that survive the renormalization procedure. The way a site i on the final renormalized lattice is related to its index on the original lattice may be realized from Fig. 1.

In order to see if anything spectacular happens to the signal-to-noise ratios, we need to know, first, where the ordered phase boundary is. In Fig. 2(a) we give the logarithm of the susceptibility [19],

$$\chi = \frac{1}{8h^2} \sum_{i=1}^{8} \langle s_i \rangle \sum_{j=1}^{N} h_j, \tag{5}$$

where h_i is the original field on the site i and N is the total number of sites in the original system. The actual ensemble averages were performed by repeating the calculations for 10,000 realizations of the randomness, summing up the results and