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# Enhancement of the stability of lattice Boltzmann methods by dissipation control

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## HIGHLIGHTS

- The stability problem arises for lattice Boltzmann methods in modelling of highly non-equilibrium fluxes.
- Dissipation control is an efficient tool to improve stability but it affects accuracy.
- We analyse the stability–accuracy problem for lattice Boltzmann methods with additional dissipation.
- We compare various methods for dissipation control: Entropic filtering, Multirelaxation methods and Entropic collisions.
- For numerical test we use the lid driven cavity; the accuracy was estimated by the position of the first Hopf bifurcation.

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## ABSTRACT

Artificial dissipation is a well known tool for the improvement of stability of numerical algorithms. However, the use of this technique affects the accuracy of the computation. We analyse various approaches proposed for enhancement of the Lattice Boltzmann Methods' (LBM) stability. In addition to some previously known methods, the Multiple Relaxation Time (MRT) models, the entropic lattice Boltzmann method (ELBM), and filtering (including entropic median filtering), we develop and analyse new filtering techniques with independent filtering of different modes. All these methods affect dissipation in the system and may adversely affect the reproduction of the proper physics. To analyse the effect of dissipation on accuracy and to prepare practical recommendations, we test the enhanced LBM methods on the standard benchmark, the 2D lid driven cavity on a coarse grid ($101\times101$ nodes). The accuracy was estimated by the position of the first Hopf bifurcation points in these systems. We find that two techniques, MRT and median filtering, succeed in yielding a reasonable value of the Reynolds number for the first bifurcation point. The newly created limiters, which filter the modes independently, also pick a reasonable value of the Reynolds number for the first bifurcation.



## 1. Introduction

Lattice Boltzmann methods (LBMs) are a type of discrete algorithm which can be used to simulate fluid dynamics and more [1–4]. One of the nicest properties of an LB scheme is that the transport component of the algorithm, advection, is exact. All of the dissipation in the discrete system then occurs due to the relaxation operation. This dissipation occurs at different orders of the small parameter, the time step [5]. The first order gives an approximation to the Navier–Stokes equations (with

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some error terms due to the discrete velocity system). The higher orders include higher space derivatives, similar to Burnett and super-Burnett type systems. (They differ from the Burnett and super-Burnett terms and may have better stability properties, see a simple case study for the Ehrenfests' collisions in Ref. [6].) The form of the dissipative terms in the macroscopic dynamics is driven by the form of the discrete equilibrium, the analogy to the Maxwell distribution, the coefficients of the dissipative terms are selected by the collision operator. In the standard and most basic relaxation operation, the single time Bhatnagar–Gross–Krook (BGK) relaxation [7], these coefficients cannot be modified independently.

At moderate to high Reynolds numbers, often corresponding to unsteady or turbulent flows, the lattice Boltzmann method using the BGK collision (LBGK) suffers from stability problems. These instabilities manifest as non-physical oscillations which rapidly increase in amplitude and cause the system to 'blow up'. These instabilities may be occur independently of whether additional techniques are implemented to improve the accuracy of such flows when they occur at low Reynolds numbers [8]. There are several existing techniques which generalize and improve on the performance of LBGK, which we give some background to. These include Multiple Relaxation Times [9–14] (MRT, Section 3), Entropic Lattice Boltzmann [15,16] (ELBM, Section 4) and entropic filtering [17–19] (Section 5).

Despite any stability benefits arising from these methods, we expect that they should each suffer a penalty in terms of additional dissipation produced in the system. In each case they work by modifying the relaxation operation in a way that increase the coefficients of the dissipative terms in the macroscopic dynamics. A part of this work is that we can compare the relative amount of dissipation added at moderate to high Reynolds numbers of these existing methods.

In this work we also attempt to combine the entropic filtering and MRT techniques by applying a type of filter to the separate modes of the MRT system. By doing so we hope to achieve the stability benefits exhibited by these methods while adding as little extra dissipation as possible.

In order to test these methods at moderate to high Reynolds numbers we used a lid-driven cavity test. This is a widely used test for low to moderate Reynolds number flows. In that case several static vortices form and the accuracy of different methods can be measured by comparing the location of the vortex centres against standard references [20,21]. For moderate to high Reynolds numbers where the flow becomes unsteady it is necessary to switch to a different measure. We use the Reynolds number at which the first Hopf bifurcation, the transition from steady to unsteady flow in the system, occurs. Any additional dissipation produced by the various methods we test will increase the Reynolds number at which this bifurcation occurs. The influence of system parameters such as aspect ratio and lid speed for the unsteady lid-driven cavity has also been studied for other Boltzmann derived numerical methods [22].

## 2. Single relaxation time LB schemes

The usual approach for deriving the single relaxation time LB systems starts from the Boltzmann Equation [1–4]

$$\partial_t f + \mathbf{v} \cdot \partial_{\mathbf{x}} f = Q(f) \tag{1}$$

where $f \equiv f(\mathbf{x}, \mathbf{v}, t)$ is a one particle distribution function over space, velocity space and time and $Q(f)$ represents the interaction between particles, sometimes called a collision operation. The nonlinear Boltzmann collision integral $Q(f)$ is substituted by the Bhatnagar–Gross–Krook (BGK) [7] operator

$$Q(f) = -\frac{1}{\tau}(f - f^{\mathrm{eq}}). \tag{2}$$

The BGK operator represents a relaxation towards the local equilibrium $f^{\mathrm{eq}}$ with rate $1/\tau$. The distribution $f^{\mathrm{eq}}$ is given by the Maxwell Boltzmann distribution,

$$f^{\mathrm{eq}} = \frac{\rho}{(2\pi\Theta)^{D/2}} \exp\left(\frac{-(\mathbf{v}-\mathbf{u})^2}{2\Theta}\right), \tag{3}$$

where $D$ is the number of spatial dimensions, $\Theta$ is the temperature in the so-called energy units in which $\Theta^{1/2} = c_s$ is the (isothermal) speed of sound.

The macroscopic quantities are available as integrals over velocity space of the distribution function,

$$\rho = \int f \,\mathrm{d}\mathbf{v}, \qquad \rho\mathbf{u} = \int \mathbf{v} f \,\mathrm{d}\mathbf{v}, \qquad \frac{1}{2}\rho\mathbf{u}^2 + \rho D\Theta = \frac{1}{2}\int \mathbf{v}^2 f \,\mathrm{d}\mathbf{v}. \tag{4}$$

A quadrature approximation to these integrals is the first ingredient to discretize this system. The second is a time integration along the discrete velocities given by the quadrature. In order to achieve a fully discrete system it is necessary to make an appropriate choice of quadrature. The set of all quadrature nodes is denoted $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. This set should be chosen such that it defines a discrete subgroup of space $\mathcal{L}$ called the lattice, which is invariant under shifts given by elements of $\mathcal{V}$, $\mathcal{L} = \mathcal{L} + \mathbf{v}_i$.

The scalar field of the population function (over space, velocity space and time) becomes a sequence of vector fields (over space) in time $f_i(\mathbf{x}, n_t\,\epsilon)$, $n_t \in \mathbb{Z}$, where the elements of the vector each correspond with an element of the quadrature.

Explicitly the macroscopic moments are given by,

$$\rho = \sum_{i=1}^{n} f_i, \qquad \rho\mathbf{u} = \sum_{i=1}^{n} \mathbf{v}_i f_i, \qquad \frac{1}{2}\left(\rho\mathbf{u}^2 + \rho D\Theta\right) = \frac{1}{2}\sum_{i=1}^{n} \mathbf{v}_i^2 f_i. \tag{5}$$