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Potts model partition functions on two families of fractal lattices

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HIGHLIGHTS

- We study two families of fractal lattices with self-similar structures.
- Their Potts model partition functions are obtained by subgraph decomposition method.
- Their spanning tree numbers and asymptotic growth constants are determined.

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ABSTRACT

The partition function of q-state Potts model, or equivalently the Tutte polynomial, is computationally intractable for regular lattices. The purpose of this paper is to compute partition functions of q-state Potts model on two families of fractal lattices. Based on their self-similar structures and by applying the subgraph-decomposition method, we divide their Tutte polynomials into two summands, and for each summand we obtain a recursive formula involving the other summand. As a result, the number of spanning trees and their asymptotic growth constants, and a lower bound of the number of connected spanning subgraphs or acyclic root-connected orientations for each of such two lattices are obtained.

1. Introduction

The Potts model [1], a generalization of the Ising model, is a valuable statistical mechanical model which models complex systems whose behaviors depend on nearest neighbor energy interactions and is used for the study of phase transitions and critical phenomena [2,3].

Let $G = (V, E)$ be a regular lattice, or more generally, a graph G with vertex set V and edge set E, and $[q] = \{1, 2, \ldots, q\}$ the set of q elements called spins. A state σ of a graph G is an assignment of a single spin $\sigma_i \in [q]$ to each vertex of the graph. The Hamiltonian and the Boltzmann weight of this state are defined respectively as

$$\mathcal{H}_\sigma = -J \sum_{e=ij\in E} \delta(\sigma_i, \sigma_j) \quad \text{and} \quad \exp(-\beta \mathcal{H}_\sigma) = \exp\left(\beta J \sum_{e=ij\in E} \delta(\sigma_i, \sigma_j)\right),$$

where δ denotes the Kronecker delta, J is the interaction energy between adjacent elements of the system. The standard (zero-field) Potts model for temperature T on G is characterized by a polynomial with two parameters: the number q of

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spins, and the nearest-neighbor coupling $v = \exp(\beta J) - 1$ (here $\beta = \frac{1}{\kappa_B T}$ and κ_B is the Boltzmann constant), i.e. the partition function

$$Z_G(q, v) = \sum_{\sigma} \exp(-\beta \mathcal{H}_\sigma). \tag{1}$$

The Fortuin–Kasteleyn representation [4] for $Z_G(q, v)$ can be written in the form

$$\begin{aligned} Z_G(q, v) &= \sum_{\sigma} \exp\left(\beta J \sum_{e=ij \in E} \delta(\sigma_i, \sigma_j)\right) \\ &= \sum_{\sigma} \prod_{e=ij\in E} (1 + v\delta(\sigma_i, \sigma_j)) \\ &= \sum_{H \subseteq G} q^{\kappa(H)} v^{|E(H)|} \end{aligned} \tag{2}$$

where $\kappa(H)$ is the number of connected components of the spanning subgraph H of G. The model is ferromagnetic when $J > 0$ and antiferromagnetic if $J < 0$.

This representation indeed defines $Z_G(q, v)$ for arbitrary complex q and v, and thus makes a connection with the celebrated Tutte polynomial [5] in combinatorics. We postpone explaining the connection to the next section. Except for its applications in statistical physics, the Tutte polynomial encodes a substantial amount of interesting information about the graph, and plays a significant role in several areas of science, for instance, combinatorics, computer science, knot theory and biology. For a thorough survey on the Tutte polynomial, we refer the reader to Refs. [6–8]. Generally, the partition function of q-state Potts model, or the Tutte polynomial, is computationally intractable [9]. In both fields of combinatorics and statistical physics, Tutte polynomials of many graphs (or lattices) have been computed by different methods, such as the transfer matrix method. See Refs. [10–20].

The number of spanning trees of a finite lattice or network G, also called the complexity of G, is a well-studied invariant and an important measure of reliability of a network [21]. See, for example, Refs. [22–29]. Among these graphs, some have self-similar structures, say, the Pseudofractal scale-free web in Ref. [27] and the Apollonian networks in Ref. [28]. In addition, the number of spanning trees is exactly the number of recurrent configurations of the Abelian sandpile model [30] and is also related to the optimal synchronization [31]. In recent years, much effort has also been taken to the study of the Tutte polynomial of self-similar graphs. For example, outplanar graphs M_n in Ref. [32], Farey graphs in Ref. [33], Sierpiński and Hanoi graphs in Ref. [34]. We also notice that the rigorous bound of the spanning-forest entropy for the square lattice was studied in Ref. [35] and was improved in Refs. [36–38].

Koch graphs (or networks) originate from Koch curves or Koch fractals. Several families of Koch networks were known with remarkable small-world properties and scale-free behavior [39–41] which can lead to a better understanding of how the underlying systems work [42]. For specific Koch networks, the mean first passage time of random walks can increase linearly with the network order [43]. Hence, Koch networks can well mimic the real networked systems in nature and society. The modified Koch graph (MKG), which is associated with the modified Koch curve (MKC), was introduced in Ref. [44] and the integrated density of states for the difference Laplacian on the MKG was discussed in Ref. [45]. The spectral properties of the MKG were further analyzed in Ref. [46]. The Austria graph is the 4-cell graph of a graph which is related to the MKC, and in which geometric properties concerning the volume growth and distances were studied in Ref. [47]. MKGs and Austria graphs are both self-similar graphs having exactly two boundary vertices in view of Ref. [48]. The number of spanning trees of the Austria graphs was determined by using the Laplacian operator in Ref. [26].

The aim of this paper is to compute the Tutte polynomial of the MKGs and Austria graphs. The approach we used is the subgraph-decomposition method which seems powerful for investigating many combinatorially and statistically interesting models, see, for instance, Refs. [24,28,32–34]. In this approach, we first analyze all spanning subgraphs of the infinite sequence of finite graphs, then we divide the Tutte polynomial into a sum of two terms by partitioning the set of the spanning subgraphs. Finally, using self-similarity, we are able to give a recursive formula for each term, involving the other term. As a result, the number of spanning trees and their asymptotic growth constants, and a lower bound of the number of connected spanning subgraphs or acyclic orientations with one predefined source for each of such two lattices are obtained.

2. The Tutte polynomial and the Potts model partition function

The Tutte polynomial, due to W.T. Tutte [5], is a two-variable polynomial satisfying a fundamental universal property with respect to the deletion–contraction reduction of a graph. The Tutte polynomial $T(G; x, y)$ of the graph G (loops and multiple edges are allowed) is defined as

$$T(G; x, y) = \sum_{H \subseteq G} (x-1)^{r(G)-r(H)} (y-1)^{n(H)}, \tag{3}$$

where the sum runs over all the spanning subgraphs H of G, and $r(G) = |V(G)| - \kappa(G)$, $n(G) = |E(G)| - |V(G)| + \kappa(G)$.

The Tutte polynomial, in a strong sense, contains every graph invariant that can be computed by deletion–contraction operations. Many polynomial invariants are evaluations of the Tutte polynomial along particular lines in the (X, Y) plane: