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Normal form transforms separate slow and fast modes in stochastic dynamical systems

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Abstract

Modelling stochastic systems has many important applications. Normal form coordinate transforms are a powerful way to untangle interesting long term macroscale dynamics from insignificant detailed microscale dynamics. We explore such coordinate transforms of stochastic differential systems when the dynamics have both slow modes and quickly decaying modes. The thrust is to derive normal forms useful for macroscopic modelling of complex stochastic microscopic systems. Thus we not only must reduce the dimensionality of the dynamics, but also endeavour to separate *all* slow processes from *all* fast time processes, both deterministic and stochastic. Quadratic stochastic effects in the fast modes contribute to the drift of the important slow modes. Some examples demonstrate that the coordinate transform may be only locally valid or may be globally valid depending upon the dynamical system. The results will help us accurately model, interpret and simulate multiscale stochastic systems.

Keywords: Stochastic dynamical systems; Multiscale modelling

1. Introduction

Normal form coordinate transformations provide a sound basis for simplifying multiscale nonlinear dynamics [e.g. 1,2]. In systems with fast and slow dynamics, a coordinate transform is sought that decouples the slow from the fast. The decoupled slow modes then provide accurate predictions for the long term dynamics. Arguably, such normal form coordinate transformations provide a much more insightful view of simplifying dynamics than other, more popular, techniques. Averaging is perhaps the most popular technique for simplifying dynamics [e.g. 3, Chapters 11–13], especially for stochastic dynamics that we explore here [e.g. 4,5]. But averaging fails in many cases. For example, consider the simple, linear, slow–fast system of stochastic differential equations (SDEs)

$$dx = \varepsilon y\, dt \quad \text{and} \quad dy = -y\, dt + dW, \tag{1}$$

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where for small parameter ε the variable $x(t)$ evolves slowly compared to the fast variable $y(t)$. Let us compare the predictions of averaging and a 'normal form' coordinate transform. First consider averaging: the fast variable y, being an Ornstein–Uhlenbeck process, rapidly approaches its limiting PDF that is symmetric in y. Then averaging the x equation leads to the prediction $d\bar{x} = 0\,dt$; that is, averaging predicts nothing happens. Yet the slow x variable must fluctuate through its forcing by the fast y. Second, and similar to illuminating coordinate transforms explored in this article, modify the x and y variables to new coordinates X and Y where

$$x = X - \varepsilon Y + \varepsilon \int_{-\infty}^{t} e^{\tau - t}\, dW_\tau \quad \text{and} \quad y = Y + \int_{-\infty}^{t} e^{\tau - t}\, dW_\tau. \tag{2}$$

In the X and Y coordinates the SDE system (1) decouples to simply

$$dX = \varepsilon\, dW \quad \text{and} \quad dY = -Y\, dt. \tag{3}$$

In these new coordinates $Y \to 0$ exponentially fast. Thus in the long term the only significant dynamics occurs in the modified slow variable X which system (3) shows undergoes a random walk. The method of averaging completely misses this random walk: true, the mean $\bar{x}$ remains at zero; but the growing spread about the mean is missed by averaging. Stochastic coordinate transforms such as (2) decouple fast and slow variables to empower us to extract accurate models for a true slow variable X. They are called 'normal form' transformations because this decoupling of stochastic dynamics is analogous to corresponding simplifications in deterministic systems [e.g. 6,7]. This article establishes useful properties for such stochastic normal form coordinate transformations in modelling multiscale nonlinear stochastic dynamical systems.

One great advantage of basing modelling upon coordinate transforms is that exactly transformed dynamics fully reproduce the original dynamics for all time and all state space. It is only when we approximate the transformed dynamics that errors occur. Consequently, modelling errors can be much better controlled.

Stochastic ODEs and PDEs have many important applications. Here we restrict attention to nonlinear SDEs when the dynamics of the SDE has both long lasting slow modes and decaying fast modes [e.g. 8]. The aim underlying all the exploration in this article is to derive normal forms useful for macroscopic modelling of stochastic systems when the systems are specified at a detailed microscopic level. Thus we endeavour to separate *all* fast time processes from *all* slow processes [e.g. 9,10]. Such separation is especially intriguing in stochastic systems as white noise has fluctuations on *all* time scales. In contrast, almost all previous approaches have been content to derive normal forms that support reducing the dimensionality of the dynamics. Here we go further than other researchers and *additionally and systematically separate fast time processes from the slow modes.*

Arnold and Imkeller [8,7] developed a rigorous body of theory to support stochastic coordinate transforms to a normal form. They comment that the normal form transformation involves anticipating the noise processes, that is, involving integrals of the noise over a fast time scale of the future. However, in contrast to the examples of Arnold and Imkeller [8] and [7, corrected], Sections 2 and 3 argue that such anticipation can be always removed from the slow modes with the result that no anticipation is required after the fast transients decay. Furthermore, Sections 2 and 3 argue that on the stochastic slow manifold (SSM) all noise integrals can be removed from terms linear in the noise to leave a slow mode system, such as the simple $dX = \varepsilon\, dW$ of the normal form (3), in which there are no fast time integrals at all. The arguments demonstrate that, except for some effects nonlinear in the noise, all fast time processes can be removed from the slow modes of a normal form of stochastic systems.

The theory of Arnold and Imkeller [8,7] applies only to finite dimensional stochastic systems. Similarly, Du and Duan's [11] theory of invariant manifold reduction for stochastic dynamical systems also only applies in finite dimensions. But many applications are infinite dimensional; for example, the discretisation of stochastic PDEs approximates an inertial manifold of stochastic dynamics [10]. Following the wide recognition of the utility of inertial manifolds [e.g. 12], Bensoussan and Flandoli [13] proved the existence of attractive stochastic inertial manifolds in Hilbert spaces. The SSMs obtained in Sections 2–4 via stochastic normal forms are examples of such stochastic inertial manifolds, albeit still in finite dimensions.

To derive a normal form we have to implement a coordinate transformation that simplifies a stochastic system. But the term 'simplify' means different things to different people depending upon how they wish to use the 'simplified' stochastic system. Our aim throughout this article is to create stochastic models that may