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Physica A 387 (2008) 71–83
PHYSICA A
www.elsevier.com/locate/physa

Multiplicative duality, q-triplet and (μ, ν, q)-relation derived from the one-to-one correspondence between the (μ, ν)-multinomial coefficient and Tsallis entropy S_q

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Received 14 February 2007; received in revised form 23 July 2007
Available online 12 September 2007

Abstract

We derive the multiplicative duality "$q \leftrightarrow 1/q$" and other typical mathematical structures as the special cases of the (μ, ν, q)-relation behind Tsallis statistics by means of the (μ, ν)-multinomial coefficient. Recently the additive duality "$q \leftrightarrow 2-q$" in Tsallis statistics is derived in the form of the one-to-one correspondence between the q-multinomial coefficient and Tsallis entropy. A slight generalization of this correspondence for the multiplicative duality requires the (μ, ν)-multinomial coefficient as a generalization of the q-multinomial coefficient. This combinatorial formalism provides us with the one-to-one correspondence between the (μ, ν)-multinomial coefficient and Tsallis entropy S_q, which determines a concrete relation among three parameters μ, ν and q, i.e., $\nu(1-\mu)+1=q$ which is called "(μ, ν, q)-relation" in this paper. As special cases of the (μ, ν, q)-relation, the additive duality and the multiplicative duality are recovered when $\nu=1$ and $\nu=q$, respectively. As other special cases, when $\nu=2-q$, a set of three parameters (μ, ν, q) is identified with the q-triplet $(q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})$ recently conjectured by Tsallis. Moreover, when $\nu=1/q$, the relation $1/(1-q_{\text{sen}}) = 1/\alpha_{\min} - 1/\alpha_{\max}$ in the multifractal singularity spectrum $f(\alpha)$ is recovered by means of the (μ, ν, q)-relation.

Keywords: Additive duality; Multiplicative duality; q-triplet; Multifractal triplet; (μ, ν, q)-relation; q-product; Tsallis entrophy; (μ, ν)-multinomial coefficient

1. Introduction

In the last two decades the so-called Tsallis statistics or q-statistics has been introduced [1] and studied as a generalization of Boltzmann–Gibbs statistics with many applications to complex systems [2,3], whose information measure is given by

$$S_q(p_1, \ldots, p_k) = \frac{1-\sum_{i=1}^{k} p_i^q}{q-1}, \tag{1}$$

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0378-4371/$ - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.physa.2007.07.074

where p_i is a probability of ith state and q is a real parameter. This generalized entropy S_q is nowadays called *Tsallis entropy* which recovers Boltzmann–Gibbs–Shannon entropy S_1 when $q \to 1$. The above entropic form (1) was first given in Refs. [4,5] from a mathematical motivation, but in 1988 [1] Tsallis first applied the above form (1) to a generalization of Boltzmann–Gibbs statistics for nonequilibrium systems through the maximum entropy principle (MaxEnt for short) along the lines of Jaynes approach [6]. Since then, many applications of (1) to the studies of complex systems with power-law behaviors have been presented using the MaxEnt as a main approach [7]. In fact, the q-exponential function appeared in the MaxEnt plays a crucial role in the formalism and applications [2,3].

For all many applications of the MaxEnt for Tsallis entropy (1), there have been missing a combinatorial consideration in Tsallis statistics until recently [8], whose ideas originate from Boltzmann's pioneering work [9] (see Ref. [10] for the comprehensive review). By means of the q-product uniquely determined by the q-exponential function [11,12] as the q-exponential law, the one-to-one correspondence between the q-multinomial coefficient and Tsallis entropy is obtained as follows [8]: for $n = \sum_{i=1}^{k} n_i$ and $n_i \in \mathbb{N}$ if $q \neq 2$,

$$\ln_q \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_q \simeq \frac{n^{2-q}}{2-q} \cdot S_{2-q}\left(\frac{n_1}{n}, \ldots, \frac{n_k}{n}\right), \tag{2}$$

where $\begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_q$ is the q-multinomial coefficient and $\ln_q$ is the q-logarithm (see Definitions 1 and 6). The above correspondence (2) obviously recovers the well-known correspondence:

$$\ln \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix} \simeq n \cdot S_1\left(\frac{n_1}{n}, \ldots, \frac{n_k}{n}\right) \tag{3}$$

when $q \to 1$. Moreover, the *additive duality* "$q \leftrightarrow 2-q$" in Tsallis statistics is presented in the form of (2). In the MaxEnt formalism for Tsallis entropy, two kinds of dualities "$q \leftrightarrow 2-q$" and "$q \leftrightarrow 1/q$" have been observed and discussed [7,13–16], but in the combinatorial formalism the *multiplicative duality* "$q \leftrightarrow 1/q$" is still missing.

In this paper, we derive the multiplicative duality "$q \leftrightarrow 1/q$" along the lines of the above correspondence (2), which introduces the (μ, v)-factorial as a generalization of the q-factorial. We apply the (μ, v)-factorial to the formulation of the (μ, v)-multinomial coefficient and (μ, v)-Stirling's formula, which results in the following correspondence: for $n = \sum_{i=1}^{k} n_i$ and $n_i \in \mathbb{N}$ if $q, v \neq 0$,

$$\frac{1}{v} \ln_\mu \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{(\mu,v)} \simeq \frac{n^q}{q} \cdot S_q\left(\frac{n_1}{n}, \ldots, \frac{n_k}{n}\right), \tag{4}$$

where $\begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_{(\mu,v)}$ is the (μ, v)-multinomial coefficient and three parameters μ, v, q satisfy the relation:

$$v(1-\mu) + 1 = q \tag{5}$$

which is called "(μ, v, q)-*relation*" throughout the paper.

Using the additive duality "$q \leftrightarrow 2-q$" in (2), (2) is rewritten by

$$\ln_{2-q} \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{2-q} \simeq \frac{n^q}{q} \cdot S_q\left(\frac{n_1}{n}, \ldots, \frac{n_k}{n}\right). \tag{6}$$

Hence the above generalized correspondence (4) is found to recover (6) when $\mu = 2-q$ and $v = 1$. As will be shown later,

$$\begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_{(\mu,1)} = \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_\mu. \tag{7}$$

The (μ, v, q)-relation (5) among three parameters μ, v, q yields the additive duality "$q \leftrightarrow 2-q$" when $v = 1$ and the multiplicative duality "$q \leftrightarrow 1/q$" when $v = q$, respectively. As other special cases of the (μ, v, q)-relation, when $v = 2-q$, it is shown that the *q-triplet* $(q_{\mathrm{sen}}, q_{\mathrm{rel}}, q_{\mathrm{stat}})$ recently conjectured by Tsallis [17] is