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Fractional dynamics of systems with long-range space interaction and temporal memory

Vasily E. Tarasov[a,b,*], George M. Zaslavsky[a,c]

[a]Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012, USA
[b]Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119992, Russia
[c]Department of Physics, New York University, 2-4 Washington Place, New York, NY 10003, USA

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Abstract

Field equations with time and coordinate derivatives of noninteger order are derived from a stationary action principle for the cases of power-law memory function and long-range interaction in systems. The method is applied to obtain a fractional generalization of the Ginzburg–Landau and nonlinear Schrödinger equations. As another example, dynamical equations for particle chains with power-law interaction and memory are considered in the continuous limit. The obtained fractional equations can be applied to complex media with/without random parameters or processes.

Keywords: Fractional derivatives; Fractional equations; Long-range interaction; Power-law memory

1. Introduction

Complex media, with its important applications and underlying microscopic processes, is far from simplicities uniform gases, liquids, or solids. The most typical features of the new physical objects and/or processes are fractality of their structure and intrinsic dynamics or kinetics. Observation of fractality of the basic processes began fairly long ago (see, for review, Refs. [1,2]). Typically the complexity of systems is linked to long-term memory, long-range interactions, non-Markovian kinetics, and particularly Levy-type processes (Levy flights) [3]. The literature on this subject is vast. Let us mention some of the most related references, where the indication of the complexity can lead, in one or another way, to the fractional description of the dynamic and/or kinetic processes with fractional time [4–6]; systems of many coupled elements [7,8]; colloidal aggregates and chemical reaction medium [9]; wave processes [10–12]; porous media [13]; quantum mechanics and quantum field theory [14–16]; plasma physics [17–19]; magnetosphere [20]; random processes and random walks [21–25]; fractional diffusion and Brownian motion [6,26,27]; weak and strong turbulence [11,28,29]; fractional kinetics and chaos theory [30] (see, for review, Refs. [31,32]).

*Corresponding author. Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119992, Russia.
Tel.: +7 95 9395989; fax: +7 95 9390397.
E-mail address: tarasov@theory.sinp.msu.ru (V.E. Tarasov).

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It seems that the basic formal tool to be applied is the description of the processes by fractional equations, i.e., by the ones that contain fractional derivatives or integrals [33–36]. The theory of derivatives of noninteger order goes back to Leibnitz, Liouville, Riemann, Grunwald, and Letnikov [33,34]. Derivatives and integrals of fractional order have found many applications in recent studies in physics because of their continually growing numerous applications.

Usually, onset of fractional derivatives (integral) is linked to different power-type asymptotic interactions or temporal memories. Depending on what kind of specific features characterize the physical object, the fractional derivative (integrals) can be with respect to time or space coordinate. In the description of particles transport, when the dynamics is chaotic, the fractional derivatives emerge in space and time simultaneously as a natural reflection of scaling properties of the phase space dynamics [30,31]. The diffusion described by the fractional equations is called anomalous. The occurrence of such derivatives could also be related to the space–time decay [37,38], i.e., to pure dynamical processes without kinetics or diffusion. Particularly it was shown in Refs. [39–43] how the long-range interaction between different oscillators can be described by fractional differential equations in the continuous medium limit. Another way to connect the fractional equations with specific dispersion laws of the media was considered in Refs. [10,11,44].

The goal of this paper is to provide a systematic approach to the onset of fractional equations as a result of existence of long-range interaction in a corresponding space and long-range time memory in the system of fields or particles depending on what kind of physical objects are considered. The notions of long-terms memory or interaction can be exactly specified by power laws in time for a memory function and power-law interaction between different elements of the medium. It is of importance to understand the conditions when the fractional derivatives (integrals) occur since it allows us to involve powerful tools of fractional calculus.

In Section 2, we consider the variation of the action functional that describes a field with memory and long-range interaction. The long-time memory and long-range interaction can be introduced through power-like kernels of the action functional. The corresponding powers are defined by the exponent α (for space) and β (for time), which in general can be fractional. The Euler–Lagrange equations lead to the equation with fractional (α,β)-derivatives. In Section 3, the obtained results are used for derivation of (α,β)-generalization of the Ginzburg–Landau and nonlinear Schrödinger equations (NSES). In Section 4, we consider chains of particles with long-range interaction and memory function. Applying the results of Section 2, we derive the continuous limit of the particle dynamics equations. In two Appendices, we provide some brief information on the Riemann–Liouville, Caputo and Riesz fractional derivatives used in paper, and n-dimensional generalization of the final fractional equations.

2. Action functional and its variation

2.1. Action functional

Let us define the action functional as

$$S[u] = \int_R \mathrm{d}^2x \int_R \mathrm{d}^2y \left(\frac{1}{2}\partial_t u(x) g_0(x,y) \partial_{t'} u(y) + \frac{1}{2}\partial_r u(x) g_1(x,y)\partial_{r'} u(y) - V(u(x),u(y))\right). \quad (1)$$

Here $x=(t,r)$, t is time, r is coordinate, and $y=(t',r')$. The integration is carried out over a region R of the two-dimensional space $\mathbb{R}^2$ to which x belong. The field $u(x)$ is defined in a two-dimensional region R of $\mathbb{R}^2$. We assume that $u(x)$ has partial derivatives

$$\partial_t u(x) = \frac{\partial u(t,r)}{\partial t}, \quad \partial_r u(x) = \frac{\partial u(t,r)}{\partial r},$$

which are smooth functions with respect to time and coordinate.

Here are three examples of this action.

(a) If

$$g_0(x,y) = -g_1(x,y) = \delta(x-y),$$
$$V(u(x),u(y)) = V(u(x))\delta(x-y), \quad (2)$$