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Multifractality attributed to dual central limit-like convergence effects

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HIGHLIGHTS

- We explain multifractal sequences through statistical convergence effects on random variables.
- The first convergence effect generates monofractal sequence with long-range correlations.
- The second convergence effect generates the variation in fractal dimension of the monofractals.
- Both convergence effects together explain the genesis of multifractal sequences.

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ABSTRACT

Multifractals can be defined as fractal systems that express a range of fractal dimensions. The origins of multifractality in time series data have conventionally been attributed to fat-tailed probability distributions, and to long-range correlations. Multifractal sequences can be generated from the eigenvalue deviations of the Gaussian unitary and orthogonal ensembles of random matrix theory. These deviations can be resolved into component monofractal sequences governed by the Tweedie compound Poisson distribution, a statistical model that expresses a variance to mean power law related to long-range correlations. Fully multifractal descriptions of these deviations can be constructed, provided that the parameter of the compound Poisson model related to fractal dimension varies in accordance with an asymmetric Laplace distribution. Both the Tweedie compound Poisson distribution and the asymmetric Laplace distribution serve as foci of convergence in limit theorems on independent and identically distributed random variables. The hypothesis that multifractal sequences can be attributed to mathematical convergence effects that have as their focus these two statistical models is proposed.

1. Introduction

Multifractality in time series data has been conventionally attributed to either broad probability distributions governing the time series or to dissimilar long-range correlations between small and large fluctuations [1]. Studies based on detrended fluctuation analysis and wavelet analysis of hydrological time series have suggested that multifractality relates to self-similar clustering on multiple time scales represented by a multiplicative cascade model [2]. In contrast, when the generalized Hurst exponent method was applied to financial time series, multifractality was related to fat-tailed distributions [3]. Despite the postulated roles of fat-tailed distributions and long-range correlations in multifractality, the origin(s) of these fat-tailed distributions and long-range correlations themselves are unclear. The genesis of multifractality thus remains a subject for continued research and conjecture.

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Wavelet analysis has also been used to study multifractality associated with the eigenvalue deviations of the Gaussian unitary and orthogonal ensembles (GUE and GOE) [4], as well as within the distribution of prime numbers [5]. The origin of multifractality in these cases was, in part, attributed to a general mathematical convergence effect that had as its focus the family of statistical distributions known as the Tweedie exponential dispersion models (EDMs) [6]. These models are characterized by a power law relationship between the variance and the mean that specifies monofractal sequences and where the power law exponent relates to fractal dimension. A complete multifractal description of such data would require a further variation in this exponent. Here, this additional variation is related to the asymmetric Laplace distribution, a distribution that belongs to a further class of Tweedie models that serve as the focus of mathematical convergence for random geometric sums. Examples of this role of the asymmetric Laplace distribution will be provided from the eigenvalue deviations of the GUE and GOE, as well as the distribution of the prime numbers.

We will begin with a brief description of the multifractal properties of the GUE and GOE. The associated eigenvalue deviations will be framed in the context of the self-similar stochastic processes used by Leland et al. to describe ethernet traffic [7]. The Tweedie EDMs, introduced next, will be shown to provide a stochastic description for self-similar processes, followed by a description of the Tweedie convergence theorem to explain the mathematical origin of self-similar processes. The asymmetric Laplace distribution is then introduced to describe the variation seen within the dimensional exponent of the Tweedie EDMs [6], and the convergence theorem governing this distribution is described. On this basis a hypothesis is proposed to explain the origin of multifractality in terms of mathematical convergence effects related to the central limit theorem (CLT) of statistics.

2. Multifractality within the GUE and GOE

The GUE and GOE consist of $N \times N$ symmetric matrices with Gaussian distributed random elements. The GUE is constructed from complex Hermitian matrices that are invariant under unitary transformation. For a $N \times N$ matrix of the GUE, with elements H_{nm}, the diagonal elements are Gaussian distributed with mean 0 and variance 1; for $m < n$ the elements are Gaussian distributed with independent real and imaginary components of mean 0 and variance 1/2; the remaining elements of the matrix are derived from the conjugate transpose $H_{mn} = H_{nm}{}^*$.

For a $N \times N$ matrix of the GOE, with elements H_{nm}, its diagonal elements are the same as for the GUE; however, for $m < n$ the GOE matrix elements are all real, Gaussian distributed, values with mean 0 and variance 1/2; the remaining elements where $n < m$ are specified by $H_{mn} = H_{nm}$. Matrices from the GOE are thus invariant under orthogonal transformations.

The ranked eigenvalues from matrices of the GUE and GOE have a distinctive behavior. As $N \to \infty$ the average density $\bar{\rho}(E)$ of ranked eigenvalues of magnitude E obeys the Wigner, or semicircular, distribution [8],

$$\bar{\rho}(E) = \begin{cases} \sqrt{2N - E^2}/\pi, & |E| < \sqrt{2N} \\ 0, & |E| > \sqrt{2N}. \end{cases} \tag{1}$$

Integration of the semicircular distribution gives the number of eigenvalues on average less than E,

$$\overline{\eta}(E) = \frac{1}{2\pi}\left(E\sqrt{2N - E^2} + 2N\sin^{-1}\left(\frac{E}{\sqrt{2N}}\right) + \pi N\right). \tag{2}$$

Individual eigenvalues E_n may be renormalized (unfolded) using

$$e_n = \bar{\eta}(E_n) = \int_{-\infty}^{E_n} \mathrm{d}E'\bar{\rho}(E'), \tag{3}$$

which separates the general trend of the sequence from the fluctuating component.

Multifractality can be demonstrated from the absolute value of the deviations between the actual and the expected cumulative number of ranked eigenvalues $E_1, E_2, \ldots, E_N$ of the GUE and GOE [4],

$$|D_n| = |n - \overline{\eta}(E_n)|\,. \tag{4}$$

Fig. 1(a) gives the values for $|D_n|$ as obtained from a 10,000 × 10,000 matrix of the GUE; Fig. 2(a) gives these values from a 10,000 × 10,000 matrix of the GOE. Both plots revealed regions where larger deviations appeared to cluster together leading to apparent cusps and troughs, themselves suggestive of multifractal singularities.

Wavelet analysis [9] was applied to further assess the multifractality of these sequences. Each sequence was separated into component wavelets derived from differentiation of the Gaussian function. A partition function was constructed; its scaling behavior was analyzed in terms of the sizes of the wavelets. For multifractals this partition function would be expected to scale with a power law of exponent $\tau(q)$. This exponent is called the multifractal scale exponent of moment order q. $\tau(q)$ specifies the multifractal spectrum; a Legendre transformation provides the corresponding singularity spectrum, $D(h) = \min_q[qh - \tau(q)]$. The $D(h)$ spectrum describes the Hausdorff (fractal) dimension where the Hölder exponent takes the value h. Monofractals would be expected to express a linear $\tau(q)$ spectrum; multifractals would express an inflection in $\tau(q)$. In contrast, the $D(h)$ spectrum of a monofractal would consist of a single point; $D(h)$ for a multifractal would resemble an inverted curve.