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Pathway model, superstatistics, Tsallis statistics, and a generalized measure of entropy

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Abstract

The pathway model of Mathai [A pathway to matrix-variate gamma and normal densities, Linear Algebra Appl. 396 (2005) 317–328] is shown to be inferable from the maximization of a certain generalized entropy measure. This entropy is a variant of the generalized entropy of order α, considered in Mathai and Rathie [Basic Concepts in Information Theory and Statistics: Axiomatic Foundations and Applications, Wiley Halsted, New York and Wiley Eastern, New Delhi, 1975], and it is also associated with Shannon, Boltzmann–Gibbs, Rényi, Tsallis, and Havrda–Charvát entropies. The generalized entropy measure introduced here is also shown to have interesting statistical properties and it can be given probabilistic interpretations in terms of *inaccuracy measure*, *expected value*, and *information content* in a scheme. Particular cases of the pathway model are shown to be Tsallis statistics [C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys. 52 (1988) 479–487] and superstatistics introduced by Beck and Cohen [Superstatistics, Physica A 322 (2003) 267–275]. The pathway model's connection to fractional calculus is illustrated by considering a fractional reaction equation.

Keywords: Generalized entropy; Tsallis entropy; Boltzmann-Gibbs entropy; Superstatistics; Entropic pathway model; Fractional calculus

1. Introduction

The fundamental problem pursued in equilibrium statistical mechanics is that given a large number of physical species, such as atoms, one wishes to know how they distribute according to some common property, e.g. velocity or energy [1]. A simple mathematical model to understand the problem is at the center of statistics and probability theory. In order to deal with applications to physical situations of interest one takes into consideration the fundamental hypothesis of equal a priori probabilities for regions in phase space of an isolated system. This hypothesis is based on our insufficient knowledge for a specification of the precise state of the physical system under consideration. This hypothesis allows us to assign systems to states that agree equally well with our knowledge of the actual condition of the system. This leads to the Boltzmann–Gibbs

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entropy, or Boltzmann principle as Einstein called it, $S = k \ln W$, where W is the thermodynamic probability which is defined as the total number of equally probable microstates corresponding to the given macrostate. The Boltzmann constant is denoted by k. The Boltzmann–Gibbs entropy is relevant for situations such that all possible states of the system are considered equally probable. If we consider such a system in contact with a thermostat then we obtain the usual Maxwell–Boltzmann distribution for the possible states by maximizing the Boltzmann–Gibbs entropy S with the normalization and energy constraints. However, in nature many systems show distributions which differ from the Maxwell–Boltzmann distribution. These are usually systems with strong autocorrelations preventing the convergence to the Maxwell–Boltzmann distribution in the sense of the central-limit theorem. Well known examples in physics are: self gravitating systems, charged plasmas, Brownian particles in the presence of driving forces, and, more generally, non-equilibrium states of physical systems [2,3]. Then it is natural to ask the question of whether non-Maxwell–Boltzmannian distributions can also be obtained from a corresponding maximum entropy principle, considering a generalized form for the entropy. For this purpose, different forms were proposed, as for instance the Tsallis entropy $S_q = (W^{1-q} - 1)/(1-q)$, where q is the entropic index, that is considered the basis for a generalization of Boltzmann–Gibbs statistical mechanics [2,3]. In the present paper we are investigating the link between entropic functionals and the corresponding families of distributions in Mathai's pathway model. We come to the conclusion that this link is also important to physically analyse fractional reaction equations in terms of probability theory.

The structure of the paper is the following: In Section 2 we introduce basic notions of Mathai's pathway model in terms of parametric families of distributions. In Section 3 we introduce a generalized entropic measure and investigate its characteristics and establish the link to parametric families of distributions in Mathai's pathway model, including Tsallis' distribution. In Section 4 we establish the link between a fractional reaction equation, its reaction coefficient considered a random variable, and Tsallis statistics and superstatistics.

2. Preliminaries for Mathai's pathway model

For practical purposes of analysing data of physical experiments and in building up models in statistical physics, we frequently select a member from a parametric family of distributions. It is often found that fitting experimental data needs a model with a thicker or thinner tail than the ones available from the parametric family, or a situation of right tail cut off [4]. The experimental data reveal that the underlying distribution is in between two parametric families of distributions. This observation either appeals to the form of the entropic functional or to the representation by a distribution function. In order to create a pathway from one functional form to another a pathway parameter is introduced and a pathway model is created in Ref. [5]. This model enables one to proceed from a generalized type-1 beta model to a generalized type-2 beta model to a generalized gamma model when the variable is restricted to be positive. More families are available when the variable is allowed to vary over the real line. Mathai [5] deals mainly with rectangular matrix-variate distributions and the scalar case is a particular case there. For the real scalar case the pathway model is the following:

$$f(x) = cx^{\gamma-1}[1 - a(1-\alpha)x^{\delta}]^{1/(1-\alpha)}, \tag{1}$$

$a > 0, \delta > 0, 1 - a(1-\alpha)x^{\delta} > 0, \gamma > 0$ where c is the normalizing constant and α is the pathway parameter. For $\alpha < 1$ the model remains as a generalized type-1 beta model in the real case. For $a = 1, \gamma = 1, \delta = 1$ we have Tsallis statistics for $\alpha < 1$ [6,7]. Other cases available are the regular type-1 beta density, Pareto density, power function, triangular and related models. Observe that (1) is a model with the right tail cut off. When $\alpha > 1$ we may write $1 - \alpha = -(\alpha - 1), \alpha > 1$ so that $f(x)$ assumes the form,

$$f(x) = cx^{\gamma-1}[1 + a(\alpha-1)x^{\delta}]^{-1/(\alpha-1)}, \quad x > 0, \tag{2}$$

which is a generalized type-2 beta model for real x. Beck and Cohen's superstatistics belong to this case (2) [8,9]. For $\gamma = 1, a = 1, \delta = 1$ we have Tsallis statistics for $\alpha > 1$ from (2). Other standard distributions coming from this model are the regular type-2 beta, the F-distribution, Lévi models and related models. When $\alpha \to 1$