Physica A 398 (2014) 229–233

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Does a particle swept by a turbulent liquid diffuse?

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HIGHLIGHTS

- We study the long time mean square displacement (MSD) of a particle swept by a random flow.
- We obtain the MSD for a family of random flows.
- For ordinary turbulence which is a member of that family we find that the swept particle super-diffuses.
- The exponent characterizing the super-diffusion is 6/5.
- We explain why it is difficult to distinguish experimentally the slight super-diffusion from diffusion.

ARTICLE INFO

Article history:
Received 20 February 2013
Received in revised form 15 October 2013
Available online 22 December 2013

Keywords:
Passive scalar
Super diffusion
Turbulence

ABSTRACT

We consider a particle swept by *ordinary* turbulent flow and find that its mean square displacement (MSD) is not linear in the traversed time, T, which would correspond to diffusion, but rather it is proportional to T raised to the power 6/5, for large times. This behavior stems from correlations of the velocity field, on length scales *much larger* than the largest scale corresponding to the inertial range. We derive our results using two different methods. The first employs an analytic self consistent path integral formulation for obtaining the statistics of the trajectory of a swept particle, given the correlations of the flow. The second involves a numerical solution of an integro-differential equation for the MSD. The conditions which make it possible to observe this behavior are discussed. A by-product of our considerations is the general large T behavior of the MSD for a large family of flows, which includes *ordinary* turbulence as a special case.

Relative diffusion of two particles carried by a turbulent flow is one of the focal points of the modern field of passive scalar turbulence [1,2]. However, the diffusion of a single particle, which is clearly not less important for assessing the risks of pollutants, has not attracted much interest. The reason is that its statistics seems to have been adequately described by an old argument of G.I. Taylor [3], who concluded that a single particle diffuses on large time scales, while at shorter times it might be ballistic. The Taylor argument is based on the assumption that the velocities of a particle swept by a liquid have a finite correlation time, τ_C. Consequently, a simple order of magnitude calculation gives its effective diffusion constant as $D = \langle \mathbf{v}^2 \rangle \tau_C$, where $\mathbf{v}$ is the velocity of the particle, $\langle \cdots \rangle$ denotes an average over all possible trajectories, and the frame of reference is chosen to render the average velocity zero. The only possible weakness in the above is the assumption of a finite correlation time, τ_C. An infinite correlation time of the Lagrangian velocity of the particle leads to super-diffusion, which is characterized by the fact that the MSD is a power law in the elapsed time with a power which is larger than one [4–7]. In fact, a family of relatively tractable models has been constructed which can yield either super-diffusion or sub-diffusion

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http://dx.doi.org/10.1016/j.physa.2013.12.020

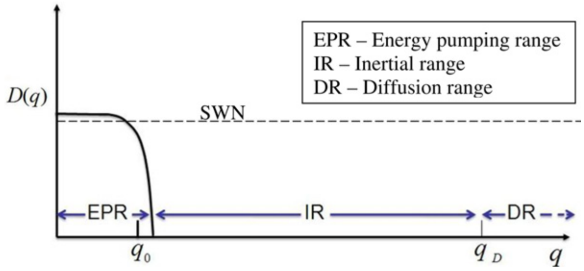

Fig. 1. Schematic description of driving noise correlations for ordinary turbulence (solid line). $D(q)$ is a short range function with range q_0. The q range relevant to the long time MSD in the case of ordinary turbulence, at very long times, is the EPR.

depending on parameters [8]. As stated in that reference, it is clear that these models, while being tractable, do not describe real world turbulence. The novel result of the present paper is that in the case of ordinary turbulence, super-diffusion with an MSD proportional to $T^{6/5}$ can indeed be observed. A new by-product of our discussion is the calculation of the MSD for a whole family of flows, which includes ordinary turbulence as a special case.

An ordinary turbulent incompressible liquid can be described by the noise driven Navier–Stokes equation [9],

$$\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P + \nu \nabla^2 \mathbf{V} + \eta \quad \text{and} \quad \nabla \cdot \mathbf{V} = 0. \tag{1}$$

The spatial Fourier transform of the noise, $\eta(\mathbf{q}, t)$ is taken to have zero mean and to have the following correlations:

$$\left\langle \eta_i(\mathbf{q}, t)\eta_j(\mathbf{p}, t')\right\rangle = \delta_{ij} D(q)\delta(\mathbf{q} + \mathbf{p})\delta(\mathbf{t} - \mathbf{t}'). \tag{2}$$

One of the objects of the theory of turbulence is to obtain the steady state time dependent correlations of the velocity field,

$$\left\langle \mathbf{V}_i(\mathbf{q}, t' + t)\mathbf{V}_j(\mathbf{p}, t')\right\rangle = \left[\delta_{ij} - \frac{\mathbf{q}_i\mathbf{q}_j}{q^2}\right]\delta(\mathbf{q} + \mathbf{p})\Phi(q, t), \tag{3}$$

which depend on the noise characteristics.

The case in which the noise is extremely infra red ($D(\mathbf{q}) \propto \delta(\mathbf{q})$), is considered in Refs. [9,10] and yields the Kolmogorov velocity field correlations as an asymptotic *small* q result. (This should not be surprising since any finite q is still infinitely large compared to zero. Thus the intermediate range includes any small non-zero q.)

The $\delta(\mathbf{q})$ in the noise correlation above is not realistic, of course, and a more realistic typical form of the correlations of the driving noise is depicted in Fig. 1, which is just meant to portray the fact that, in contrast to the previous idealization, energy is mainly pumped into the system over a small but *finite* range of "momenta", q_0. It is clear that it is possible to have experimental situations in which $q_0 L \gg 1$, where L is the linear size of the system. In such cases, the fact that the EPR (Energy Pumping Range) is finite is relevant. Ordinary turbulence is characterized by having three well separated regions: The low momentum range is the EPR. The intermediate momentum range (the inertial range) is bound from below by q_0 and from above by $q_D = \left\langle \mathbf{V}^2\right\rangle^{1/2}/\nu$, which indicates the "momentum" above which dissipation of mechanical energy into heat due to viscosity becomes effective. The usual situation is that $q_D/q_0 \gg 1$ (Fig. 1). The last relation is the one that allows observation of the characteristic exponents describing steady state turbulence. The study of passive scalars focuses usually on the inertial range and to some extent also on the dissipative range [11]. Velocity field correlations in the inertial and energy dissipation (ID) ranges are relevant, however, only to the relatively short time motion of a swept particle. The long time behavior of the trajectory of such a particle involves also very large spatial scales and is consequently mostly affected by very large scale disturbances of the velocity field. Namely, the "momentum" range relevant to the asymptotic behavior of the MSD, Δ^2, is not the inertial range but rather the energy pumping range. (This would not be the case if the energy input is concentrated only at $q = 0$ [9,10].) The necessary velocity field correlations involve a small region deep in the EPR, over which $D(q)$ can be taken to be constant. The necessary correlations have been already obtained by two of us (MS and SFE) in the general context of non-linear stochastic systems [10]. One of the problems considered in that reference is that of an incompressible liquid driven by spatial white noise (SWN). (SWN is characterized by $D(q)$ which is constant in q). The basic reason that we can use the velocity field correlation obtained for SWN for ordinary turbulence deep in the energy pumping range is that the driving noise in that q region does not look different from spatial white noise.

Clearly, the correlations of high momentum modes ($q > q_0$) of the velocity fields in ordinary turbulence are very different from the corresponding correlations of SWN systems. The point however is how those correlations feed into the equation describing the low momentum ($q < q_0$) correlations. It turns out that the effective low momentum theories, obtained by taking into account the different high momentum behavior of ordinary turbulence and SWN turbulence, differ only in their effective low momentum viscosities. High momentum behavior when fed into the equation for low momentum velocity correlations, only renormalizes the viscosity. The reader interested in the technical points raised here will find the discussion of a very similar problem in [12].