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A model for stock returns and volatility

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HIGHLIGHTS

- We show that historic volatility is best described by the generalized inverse gamma distribution.
- We show that historic stock returns are best described by the generalized Student's distribution.
- We discuss stochastic stock and volatility models that produce these distributions.
- We obtain the mean and the variance of relaxation times on approach to steady state distributions.
- We examine $1/f$ noise in volatility and stock returns.

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ABSTRACT

We prove that Student's t-distribution provides one of the better fits to returns of S&P component stocks and the generalized inverse gamma distribution best fits VIX and VXO volatility data. We further prove that stock returns are best fit by the product distribution of the generalized inverse gamma and normal distributions. We find Brown noise in VIX and VXO time series and explain the mean and the variance of the relaxation times on approach to the steady-state distribution.

1. Introduction

The generalized inverse gamma (GIGa) function (Appendix A) belongs to a family of distributions (Appendix B), which includes inverse gamma (IGa), lognormal (LN), gamma (Ga) and generalized gamma (GGa). The remarkable property of GIGa is its power-law tail; for a general three-parameter case, $\mathrm{GIGa}(x; \alpha, \beta, \gamma) \propto x^{-1-\alpha\gamma}$, $x \to \infty$. GIGa emerges as a steady state distribution in a number of problems, from a network model of economy [1,2], to ontogenetic mass growth [3], to response times in human cognition [4]. This common feature can be traced to a birth–death phenomenological model subject to stochastic perturbations (Appendix C). Here we argue that the GIGa distribution best describes the stock volatility distribution and the product distribution (Appendix D) of GIGa and normal (N) distributions, GIGa $*$ N, best describes the stock return distribution.

Numerically, we used the maximum likelihood method to determine the best parameters for each of the distributions in the above family of distributions[1] and found that GIGa provides the best fit for VIX and VXO volatility data [5–7]. We also found that among product distributions of the above family with normal distribution, GIGa's product with N gives the best fit to the stock return distribution. Furthermore, among the better GIGa $*$ N fits are those with $\gamma \approx 2$.

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[1] With the exception of GGa.

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In general, product distribution GIGa $*$ N has $|x|^{-1-\alpha\gamma}$ tails [left and right] and, for $\gamma = 2$ in particular, the product distribution GIGa$(\alpha, \beta, 2) * N$ for stock returns is the generalized Student's t-distribution, which has $|x|^{-1-2\alpha}$ tails [8–10]. Accordingly, our starting point is the geometric Brownian motion model of stock price [11,12], where the steady-state distribution of stock returns is given by the product distribution of volatility and normal distributions. Furthermore, the instantaneous variance of volatility (or square stochastic volatility—the terms used interchangeably) is described by the Nelson diffusion limit (NDL) of the GARCH(1, 1) model of stock volatility [13,14], whose stochastic term is uncorrelated from that in the equation for stock price; in the steady state, it is distributed as IGa, that is GIGa with $\gamma = 1$.

It should be noted that the product distributions of lognormal [16], gamma [22] and inverse gamma [8] distributions with normal distribution had been previously considered. Our key observation is that these distributions belong to a *family* of distributions and that it is this family of distributions that should be studied. Our work thus unifies and generalizes previous important contributions on stock returns.

This paper is organized as follows. In Section 2, we discuss stochastic stock and volatility models. In Section 3, we fit VIX and VXO, including direct evaluation of their power law tail exponents by the log–log plot. We also address Brown noise observed in the VIX/VXO time series. In Section 4, we discuss numerical results of fitting returns of S&P component stocks[2] based on log-likelihood and discuss white noise in stock return series. In Section 5, we summarize our key findings.

2. Stochastic stock and volatility models

The widely accepted equation for stock price is given by

$$\frac{dS}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}W_1 \tag{1}$$

where μ is a constant and σ volatility. The equation for the instantaneous volatility variance (square volatility) can be written in the following general form:

$$\mathrm{d}V = \tilde{f}(V)\mathrm{d}t + \tilde{g}(V)\mathrm{d}W_2. \tag{2}$$

Here $\mathrm{d}W_1$ and $\mathrm{d}W_2$ are Wiener processes correlated by $\langle \mathrm{d}W_1 \mathrm{d}W_2 \rangle = \rho \mathrm{d}t$. Substituting $V = \sigma^2$ and using Ito calculus, we obtain the volatility equation

$$\mathrm{d}\sigma = f(\sigma)\mathrm{d}t + g(\sigma)\mathrm{d}W_2. \tag{3}$$

The Fokker–Planck equation for the distribution function of σ, $P(\sigma, t)$, is given by

$$\frac{\partial}{\partial t}P(\sigma, t) = \frac{1}{2}\frac{\partial^2}{\partial \sigma^2}[g^2(\sigma)P(\sigma, t)] - \frac{\partial}{\partial \sigma}[f(\sigma)P(\sigma, t)]. \tag{4}$$

It has a stationary (steady-state) solution given by

$$P(\sigma) = \frac{2}{g^2}\exp\left(\int \frac{2f}{g^2}\mathrm{d}\sigma\right). \tag{5}$$

In what follows, we shall assume that $\mathrm{d}W_1$ and $\mathrm{d}W_2$ are uncorrelated, that is $\rho = 0$. A number of possible forms of $f(\sigma)$ and $g(\sigma)$ are discussed in Appendix E; see also [16]. Here we concentrate on one particular form

$$\mathrm{d}\sigma = J(\theta\sigma^{1-\gamma} - \sigma)\mathrm{d}t + \Sigma\sigma \mathrm{d}W_2. \tag{6}$$

The stationary (see Appendix F for discussion of relaxation times) solution of this equation is given by a three-parameter GIGa distribution

$$\mathrm{GIGa}(x; \alpha, \beta, \gamma) = \frac{\gamma}{\beta\Gamma(\alpha)} e^{-\left(\frac{\beta}{x}\right)^{\gamma}} \left(\frac{\beta}{x}\right)^{1+\alpha\gamma} \tag{7}$$

as

$$P(\sigma) = \mathrm{GIGa}\left(\sigma; \left(1 + \frac{2J}{\Sigma^2}\right)\gamma^{-1}, \left(\theta\frac{2J}{\Sigma^2}\gamma^{-1}\right)^{1/\gamma}, \gamma\right), \tag{8}$$

where the parameter θ can be expressed using the mean $\overline{\sigma}$ as

$$\theta = \frac{\gamma\Sigma^2}{2J}\left[\frac{\overline{\sigma}\Gamma\left(\left(1 + \frac{2J}{\Sigma^2}\right)\gamma^{-1}\right)}{\Gamma\left(\frac{2J}{\Sigma^2}\gamma^{-1}\right)}\right]^{\gamma}. \tag{9}$$

[2] DJIA components are fitted in the same fashion leading to identical conclusions, which is described elsewhere.