Available online at www.sciencedirect.com
ScienceDirect
Physica A 372 (2006) 238–242
PHYSICA A
www.elsevier.com/locate/physa

Sensitivity function and entropy increase rates for z-logistic map family at the edge of chaos

Ahmet Celikoglu, Ugur Tirnakli*

Department of Physics, Faculty of Science, Ege University, 35100 Izmir, Turkey

Available online 8 September 2006

Abstract

It is well known that, for chaotic systems, the production of relevant entropy (Boltzmann–Gibbs) is always linear and the system has strong (exponential) sensitivity to initial conditions. In recent years, various numerical results indicate that basically the same type of behavior emerges at the edge of chaos if a specific generalization of the entropy and the exponential are used. In this work, we contribute to this scenario by numerically analyzing some generalized nonextensive entropies and their related exponential definitions using z-logistic map family. We also corroborate our findings by testing them at accumulation points of different cycles.

Keywords: Nonlinear dynamics; Nonextensivity

In nonlinear dynamics, it is well known that the chaotic systems have an exponential sensitivity to initial conditions, characterized by the sensitivity function (for one-dimensional case)

$$\xi(t) \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} \tag{1}$$

(where $\Delta x(t)$ is the distance, in phase space, between two copies at time t) to diverge as $\xi(t) = \exp(\lambda_1 t)$, where λ_1 is the standard Lyapunov exponent [1]. On the other hand, for the marginal case, where $\lambda_1 = 0$, the form of the sensitivity function could be a whole class of functions. One of the candidates is a power-law behavior, which can be characterized by an appropriate generalization of exponentials, namely $\xi(t) = \widetilde{\exp}(\lambda t)$, where λ is the generalized Lyapunov exponent. Therefore, $\lambda > 0$ and $\lambda < 0$ cases correspond to weak sensitivity and weak insensitivity to initial conditions, respectively. This, in fact, constitutes a unified framework since the generalized exponentials include the standard one as a special case for an appropriate choice of related parameter.

The other concept that we focus in this work is the entropy production. For a chaotic system, the Kolmogorov–Sinai (KS) entropy K_1 is defined as the increase, per unit time, of the standard Boltzmann–Gibbs entropy and it is basically related to the standard Lyapunov exponents through the Pesin

*Corresponding author.
E-mail address: ugur.tirnakli@ege.edu.tr (U. Tirnakli).

0378-4371/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2006.08.008

identity, which states that $K_1 = \lambda_1$ if $\lambda_1 > 0$ and $K_1 = 0$ otherwise. Here, it is worth mentioning that the KS entropy is basically defined in terms of a single trajectory in phase space, using a symbolic representation of the regions of a partitioned phase space (see Ref. [1]). However, it appears that, in most cases, this definition can be replaced by one based on an ensemble of initial conditions, which is the version we use herein. In the framework of this version, it has already been shown that the statistical definition of entropy production rate exhibits a close analogy to the production rate of thermal entropy and practically coincides with the KS entropy in chaotic systems [2]. In recent years, there have been efforts on extending this picture to dynamical systems at their marginal points (like chaos threshold) by using a generalized entropic form, which allows us to define a generalized KS entropy K. From this definition, one can also conjecture a Pesin-like identity as $K = \lambda$, which recovers both the chaotic and critical (i.e., chaos thresholds) cases since it includes standard Pesin identity as a special case [3].

As a whole, this unified framework has been numerically verified firstly for the logistic map [3,4] using the Tsallis entropy $S_q \equiv (1 - \sum_{i=1}^{W} p_i^q)/(q-1)$ [5], which grows linearly for a special value of entropic index q, which is $q_{sen} \simeq 0.24$; whereas the asymptotic power-law sensitivity to initial conditions is characterized by the generalized exponential $\widetilde{\exp}(x) = \exp_q(x) = [1 + (1-q)x]^{1/(1-q)}$ with the same value of the entropic index. After these works on the logistic map, numerical evidences supporting this framework came also from the studies of other low-dimensional dynamical systems [6]. Besides these numerical investigations, analytical treatment of the subject is also available recently in a series of paper by Baldovin and Robledo [7]. Very recently, a similar analysis has been performed for the z-logistic map family, but this time, using ensemble-averaged initial conditions distributed uniformly over the entire available phase space [8,9]. The most important outcome of this analysis is another numerical verification of the coincidence of the entropic indices coming from the sensitivity function and entropy production rates (although with a different value $q_{sen}^{av} \simeq 0.36$), which consequently broadens the validity of the Pesin-like identity.

Finally, here we should mention a recent work of Tonelli et al. [10], where they demonstrate that the above-mentioned framework is even more general by making use of a two-parameter family of logarithms [11]

$$\widetilde{\log}(\xi) = \frac{\xi^{\alpha} - \xi^{-\beta}}{\alpha + \beta}, \tag{2}$$

where α (β) characterizes the large (small) argument asymptotic behavior. From this wide class, they analyzed four interesting one-parameter cases, namely, (i) the Tsallis [5] logarithm: $\alpha = 1 - q$ and $\beta = 0$; (ii) the Abe [12] logarithm: $\alpha = 1 - q$ and $\beta = \alpha/(1+\alpha)$; (iii) the Kaniadakis [13] logarithm: $\alpha = \beta = \kappa = 1 - q$; (iv) the γ logarithm: $\alpha = 1 - q$ and $\beta = (1-q)/2$. Their analysis consists of studying the sensitivity function and the entropy increase rates for the logistic map at the edge of chaos. Obviously, for the corresponding entropy in each case, one needs to use

$$S(t) = \sum_{i=1}^{W} p_i(t) \widetilde{\log}\left(\frac{1}{p_i(t)}\right) \tag{3}$$

from where the entropy production rates in time can be calculated. Their numerical results clearly verified that, for the logistic map, the relevant value of the entropic index is $q_{sen}^{av} \simeq 0.36$ not only for the Tsallis case but also for the others as well, and moreover that the Pesin-like identity is also present for all cases (with different numerical values for each case).

The aim of the present effort is to check the validity of the above-mentioned picture making use of the z-logistic map family. Moreover, the effect of different cycles on this validity is tested by analyzing four distinct cycles.

The model system that we use in our analysis is the z-logistic map family defined as $x_{t+1} = 1 - a|x_t|^z$, where $z > 1$, $0 < a \leqslant 2$, $|x_t| \leqslant 1$, $t = 0, 1, 2, \ldots$. It is easily seen that $z = 2$ case corresponds to the standard logistic map. Firstly, we study the sensitivity to initial conditions at the edge of chaos using the sensitivity function given in Eq. (1). From its definition, for the calculation, we proceed with considering two initially very close points, which makes for example $\Delta x(0) = 10^{-12}$ and then at each time step we numerically calculate the