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Stretched exponentials from superstatistics

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Abstract

Distributions exhibiting fat tails occur frequently in many different areas of science. A dynamical reason for fat tails can be a so-called superstatistics, where one has a superposition of local Gaussians whose variance fluctuates on a rather large spatio-temporal scale. After briefly reviewing this concept, we explore in more detail a class of superstatistics that hasn't been subject of many investigations so far, namely superstatistics for which a suitable power β^η of the local inverse temperature β is χ^2-distributed. We show that $\eta > 0$ leads to power-law distributions, while $\eta < 0$ leads to stretched exponentials. The special case $\eta = 1$ corresponds to Tsallis statistics and the special case $\eta = -1$ to exponential statistics of the square root of energy. Possible applications for granular media and hydrodynamic turbulence are discussed.

Keywords: Fluctuating temperature; Superstatistics; Fat tails

1. Introduction

Nonextensive statistical mechanics [1–4] was originally developed as an equilibrium formalism, but most physical applications of this formalism actually occur for typical nonequilibrium situations. Sometimes these nonequilibrium situations are described by a fluctuating parameter β, which may, for example, be the inverse temperature. Alternatively, β may be an effective friction constant, a changing mass parameter, a changing amplitude of Gaussian white noise, a fluctuating local energy dissipation or simply a local inverse variance parameter extracted from a time series. The fluctuations of β induce a superposition of different statistics on different time scales, in short a superstatistics [5–23]. The stationary probability distributions of superstatistical systems typically exhibit much broader tails than a Gaussian distribution. These tails can decay e.g., with a power-law, or as a stretched exponential, or in an even more complicated way [8]. Which type of tails are produced depends on the probability distribution $f(\beta)$ of the parameter β. Recent applications of the superstatistics concept include a variety of physical systems. Examples are Lagrangian [24–27] and Eulerian turbulence [28–30], defect turbulence [31], atmospheric turbulence [32,33], cosmic ray statistics [34], solar flares [35], solar wind statistics [36], networks [37,38], random matrix theory [39], and mathematical finance [40–42].

If β is distributed according to a particular probability distribution, the χ^2-distribution, then the corresponding marginal stationary distributions of the superstatistical system obtained by integrating over all

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β are given by the generalized canonical distributions of nonextensive statistical mechanics [1–4]. For other distributions of the intensive parameter β, one ends up with more complicated statistics.

In this paper, after briefly reviewing the superstatistics concept, we explore a rather general case which may be of relevance to many practical applications. We consider the case of a superstatistics where β^η, i.e., β to some power η, is χ^2-distributed, where η is some arbitrary parameter. The case $\eta = 1$ is fully understood: it leads to Tsallis statistics and asymptotic power-law decay of the marginal distributions obtained by integrating over all β. However, the other values of η are interesting as well, and will be explored in more detail here. For general $\eta > 0$ we obtain asymptotic power law decay, though the resulting statistics is slightly different from Tsallis statistics (only $\eta = 1$ leads exactly to Tsallis statistics). For $\eta < 0$ one obtains tails that asymptotically decay as stretched exponentials. The special case $\eta = -1$ corresponds to exponential tails of the square root of energy. We will provide some arguments (based on the ordinary central limit theorem) why nonequilibrium systems with many degrees of freedom often lead to one of the superstatistics described above.

2. Various types of superstatistics

It is well known that for the canonical ensemble the probability to observe a state with energy E is given by

$$p(E) = \frac{1}{Z(\beta)} \rho(E)\, \mathrm{e}^{-\beta E} \tag{1}$$

$\mathrm{e}^{-\beta E}$ is the Boltzmann factor, $\rho(E)$ is the density of states and $Z(\beta)$ is the normalization constant of $\rho(E)\,\mathrm{e}^{-\beta E}$. For superstatistical systems, one generalizes this approach by assuming that β is a random variable as well. Indeed, a driven nonequilibrium system is often inhomogeneous and consist of many spatial cells with different values of β in each cell. The cell size is effectively given by the correlation length of the continuously varying β-field. If we assume that each cell reaches local equilibrium very fast, i.e., the associated relaxation time is short, then in the long-term run the stationary probability distributions $p(E)$ arise as the following mixture of Boltzmann factors:

$$p(E) = \int_0^\infty f(\beta) \frac{1}{Z(\beta)} \rho(E)\, \mathrm{e}^{-\beta E}\, \mathrm{d}\beta. \tag{2}$$

Without restriction of generality, we may absorbe the factor $1/Z(\beta)$ into the function $f(\beta)$, i.e., define $\tilde{f}(\beta) = f(\beta)/Z(\beta)$ and rename $\tilde{f} \to f$. Also, for reasons of simplicity we may just assume $\rho(E) = 1$, keeping in mind that the most general case may correspond to a different density of states. The result is an effective distribution

$$p(E) \sim \int_0^\infty f(\beta)\, \mathrm{e}^{-\beta E}\, \mathrm{d}\beta \tag{3}$$

given essentially by the Laplace transform of $f(\beta)$.

The simplest dynamical example of a superstatistical system is a Brownian particle of mass m moving through a changing environment in d dimensions. For its velocity $\vec{v}$ one has the local Langevin equation

$$\dot{\vec{v}} = -\gamma \vec{v} + \sigma \vec{L}(t). \tag{4}$$

($\vec{L}(t)$: d-dimensional Gaussian white noise, γ: friction constant, σ: strength of noise) which becomes superstatistical if the parameter $\beta := (2/m)\gamma/\sigma^2$ is regarded as a random variable as well. In a fluctuating environment β may vary from cell to cell on a large spatio-temporal scale. Of course, for this example the energy E is just kinetic energy $E = \frac{1}{2} m \vec{v}^2$. In each cell of constant β the local stationary velocity distribution is Gaussian,

$$p(\vec{v}|\beta) = \left(\frac{\beta}{2\pi}\right)^{d/2} \mathrm{e}^{-1/2 \beta m \vec{v}^2}, \tag{5}$$