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Relaxation dynamics of the Kuramoto model with uniformly distributed natural frequencies

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HIGHLIGHTS

- Kuramoto oscillators with uniform frequency distribution show step like relaxation.
- The metastable state is dependent on the realization of the frequency distribution.
- The metastability in the dynamics of the Kuramoto model is a finite size effect.

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ABSTRACT

The Kuramoto model describes a system of globally coupled phase-only oscillators with distributed natural frequencies. The model in the steady state exhibits a phase transition as a function of the coupling strength, between a low-coupling incoherent phase in which the oscillators oscillate independently and a high-coupling synchronized phase. Here, we consider a uniform distribution for the natural frequencies, for which the phase transition is known to be of first order. We study how the system close to the phase transition in the supercritical regime relaxes in time to the steady state while starting from an initial incoherent state. In this case, numerical simulations of finite systems have demonstrated that the relaxation occurs as a step-like jump in the order parameter from the initial to the final steady state value, hinting at the existence of metastable states. We provide numerical evidence to suggest that the observed metastability is a finite-size effect, becoming an increasingly rare event with increasing system size.

Coupled oscillators that have their natural frequencies distributed according to a given distribution, for example, a Gaussian, a Lorentzian, or a uniform distribution, often exhibit collective synchronization in which a finite fraction of the oscillators oscillates with a common frequency. Examples include groups of fireflies flashing in unison [1,2], networks of pacemaker cells in the heart [3,4], superconducting Josephson junctions [5,6], and many others. Understanding the nature and emergence of synchronization from the underlying dynamics of such systems is an issue of great interest. A paradigmatic model in this area is the so-called Kuramoto model involving globally-coupled oscillators [7]. Although studied extensively in the past, the model continues to raise new questions, and has been a subject of active research; for reviews, see Ref. [8,9].

One issue that has been explored in recent years, and is also the focus of this paper, concerns the Kuramoto model with uniformly distributed natural frequencies. In this case, it is known that in the limit of infinite system size, where size refers to the number of oscillators, the system in the steady state undergoes a first-order phase transition across a critical coupling threshold K_c, from a low-coupling incoherent phase to a high-coupling synchronized phase. For values of the coupling constant slightly higher than K_c, non-trivial relaxation dynamics has been reported, based on numerical

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simulations of large systems [10,11]. Namely, it has been shown that initial incoherent states while evolving in time get stuck in metastable states before attaining synchronized steady states. This phenomenon has been demonstrated by the temporal behavior of the order parameter characterizing the phase transition, which shows a relaxation from the initial value of the order parameter to its final steady state value in step-like jumps. An aspect of the Kuramoto model which is of interest and has been explored in some detail concerns finite-size effects [12,13], which may have important consequences, for example, for $K < K_c$, in stabilizing the incoherent state which in the limit of infinite size is known to be linearly neutrally stable [14]. In this context, it is important to investigate whether the metastable states mentioned above may be attributed to finite-size effects. In this paper, we systematically study this phenomenon of step-like relaxation. We provide numerical evidence to suggest that the observed metastability is indeed a finite-size effect, becoming an increasingly rare event with increasing system size.

The Kuramoto model consists of N phase-only oscillators labeled by the index $i = 1, 2, \ldots, N$. Each oscillator has its own natural frequency ω_i distributed according to a given probability density $g(\omega)$, and is coupled to all the other oscillators. The phase of the oscillators evolves in time according to Ref. [7]

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \tag{1}$$

where θ_i, the phase of the ith oscillator, is a periodic variable of period 2π, and $K \geq 0$ is the coupling constant.

The Kuramoto model has been mostly studied for a unimodal $g(\omega)$, i.e., one which is symmetric about the mean frequency $\omega = \Omega$, and which decreases monotonically and continuously to zero with increasing $|\omega - \Omega|$ [8,9]. Then, it is known that in the limit $N \to \infty$, the system of oscillators in the steady state undergoes a continuous transition at the critical threshold $K_c = 2/\pi g(0)$. For $K < K_c$, each oscillator tends to oscillate independently with its own natural frequency. On the other hand, for $K > K_c$, the coupling synchronizes the phases of the oscillators, and in the limit $K \to \infty$, they all oscillate with the mean frequency Ω. The degree of synchronization in the system at time t is measured by the complex order parameter

$$\mathbf{r}(t) = r(t)\mathrm{e}^{\mathrm{i}\psi(t)} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j(t)}, \tag{2}$$

with magnitude $r(t)$ and phase $\psi(t)$, in terms of which the time evolution (1) may be written as

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + Kr(t)\sin(\psi(t) - \theta_i). \tag{3}$$

Here $r(t)$ with $0 \leq r(t) \leq 1$ measures the phase coherence of the oscillators, while $\psi(t)$ gives the average phase. When K is smaller than K_c, the quantity $r(t)$ while starting from any initial value relaxes in the long-time limit to zero, corresponding to an incoherent phase in the steady state. For $K > K_c$, on the other hand, $r(t)$ grows in time to asymptotically saturate to a non-zero steady state value $r_{\text{st}} = r_{\text{st}}(K) \leq 1$ that increases continuously with K. The relaxation of $r(t)$ to steady state is exponentially fast for $K > K_c$. For $K < K_c$, however, the nature of relaxation depends on $g(\omega)$. When $g(\omega)$ has a compact support, $r(t)$ while starting from any initial value decays to zero more slowly than any exponential as $t \to \infty$ [15]. When $g(\omega)$ is supported on the whole real line, $r(t)$ as a function of time is known only in particular cases. For example, for a Lorentzian $g(\omega)$, and a specific initial condition, $r(t)$ decays exponentially to zero [15]. For other choices of $g(\omega)$ in this class and for other initial conditions, the dependence of $r(t)$ on time is not known analytically, and it has been speculated that $r(t)$ is a sum of decaying exponentials [15].

In the limit $N \to \infty$, the state of the oscillator system at time t is described by the probability distribution $f(\theta, t, \omega)$ that gives for each natural frequency ω the fraction of oscillators with phase θ at time t. The time evolution of $f(\theta, t, \omega)$ satisfies the continuity equation for the conservation of the number of oscillators with natural frequency ω, and is given by a nonlinear partial integro-differential equation [8]. Recent analytical studies for a unimodal $g(\omega)$ (specifically, a Lorentzian) and for two different bimodal $g(\omega)$'s (given by a suitably defined sum and difference of two Lorentzians) demonstrated by considering a restricted class of $f(\theta, t, \omega)$, and by employing an ansatz due to Ott and Antonsen that the time evolution in terms of the integro-differential equation may be exactly reduced to that of a small number of ordinary differential equations (ODEs) [16–18]. Interestingly, the ODEs for the reduced system contain the whole spectrum of dynamical behavior of the full system. The Ott–Antonsen ansatz has also been applied to various globally and nonlinearly coupled oscillators with uniformly distributed frequencies [19].

A uniform $g(\omega)$ with a compact support does not qualify as a unimodal distribution. In this case, it is known that in the limit $N \to \infty$, the Kuramoto model in the steady state exhibits a first-order phase transition between an incoherent and a synchronized phase at the critical coupling $K_c = 2/\pi g(0)$ [20]. For large N, numerical studies of the relaxation of an initial state with uniformly distributed phases have demonstrated that for values of K around K_c in the supercritical regime, the process occurs as a step-like jump in $r(t)$ from its initial to the steady state value. One may interpret this behavior as suggesting the existence of metastable states in the system [10,11]. Our motivation is to investigate the implication of the existence of the step-like relaxation, and whether such relaxation can be seen only in finite-sized systems.

We performed extensive numerical simulations involving integration of Eq. (3) by a 4th-order Runge–Kutta algorithm. We considered a system of $N = 1000$ oscillators, with the ω_i's independently and uniformly distributed in $[-2, 2]$, so