Physica A 390 (2011) 392–397

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

On co-evolution and the importance of initial conditions

Renaud Lambiotte [a,*], Juan Carlos González-Avella [b]

[a] Institute for Mathematical Sciences, Imperial College London, 53 Prince's Gate, SW7 2PG London, UK

[b] IFISC, Instituto de Física Interdisciplinary Sistemas Complejos (CSIC-UIB), E-07122 Palma de Mallorca, Spain

ARTICLE INFO

Article history:
Received 4 August 2010
Available online 25 September 2010

Keywords:
Co-evolution
Complex networks

ABSTRACT

We present a generic threshold model for the co-evolution of the structure of a network and the binary state of its nodes. We focus on regular directed networks and derive equations for the evolution of the system toward its absorbing state. It is shown that the system displays a transition from a connected phase to a fragmented phase, and that this transition is driven by the initial configuration of the system, as different initial conditions may lead to drastically different final configurations. Computer simulations are performed and confirm the theoretical predictions.

1. Introduction

The relation between a cause and its effect is usually abrupt in complex systems, in the sense that a small change in the neighborhood of a subsystem may (or may not) trigger its reaction. This mechanism is at the heart of many models of self-organized criticality [1] where a cascade starts when the system has been frustrated beyond some threshold, e.g. the angle of a sand pile, but also in models for the diffusion of ideas in social networks [2–5] where the adoption of a new idea requires simultaneous exposure to multiple active acquaintances, and in integrate-and-fire neuron dynamics [6] where the voltage on a single neuron increases until a specified threshold is reached and it suddenly fires by emitting an action potential, thereby quickly returning to its reference. These types of model consist in cascading propagations on a fixed topology, i.e., a network of some sort, until a frozen configuration is reached, but they do not incorporate the well-known feedback existing between network topology and dynamics [7–16], namely that the topology itself may reorganize when it is not compatible with the state of the nodes. This reorganisation may originate from homophily and social balance in social networks or synaptic plasticity in neuron dynamics.

The purpose of this paper is to clarify the important role of initial conditions for such models where the nodes' states co-evolve with the network architecture. Let us describe this effect for the generic co-evolutionary threshold dynamics defined as follows (CTD). We present its ingredients in terms of the diffusion of opinions in social networks [17,18] while keeping in mind that the model is applicable to more general systems. The system is made of a social network of interaction, whose N nodes are endowed with a binary opinion s, $+$ or $-$. The dynamics is driven by the threshold ϕ, such that $0 \leq \phi \leq 1$ and, in most cases of interest, $\phi > 1/2$. At each step, a randomly selected node i evaluates the opinion of its k_i neighbours (see Fig. 1). Let ϕ_i be the fraction of neighbours disagreeing with i. If $\phi_i \leq \phi$, node i breaks the links toward those disagreeing neighbours and rewires them to randomly selected nodes. If $\phi_i > \phi$, i adopts the state of the majority. By construction, the dynamics perdures until consensus, i.e., all agents having the same opinion, has been attained in the whole system or in disconnected components. This absorbing state obviously depends on the threshold ϕ but also, as we will discuss in detail below, on its initial condition, as some classes of initial configurations will lead to fragmentation while others will lead to global (connected) consensus.

* Corresponding author. Tel.: +44 7503443917.
E-mail address: r.lambiotte@imperial.ac.uk (R. Lambiotte).

0378-4371/$ – see front matter
doi:10.1016/j.physa.2010.09.024

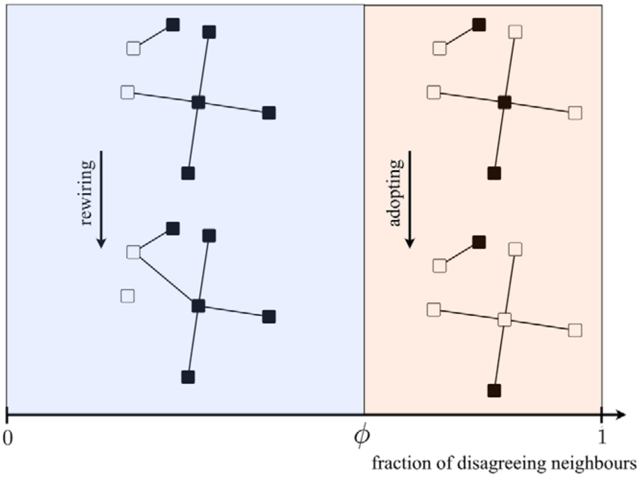

Fig. 1. Update process of CTD for two different configurations of neighbours of a randomly selected node (surrounded). When one out of four neighbours is in a different state, the central node breaks its links and creates a new link to a randomly chosen node. When three out of four neighbours are in a different state, the threshold ϕ is exceeded and the central node thus adopts the majority state.

A complete analysis of CTD requires extensive computer simulations, which is not the objective of this paper. We will instead focus on a simplified version of the model that can be studied analytically and pinpoint the key mechanisms responsible for its behaviour [19]. This model is introduced in Section 2 and studied analytically and numerically in Section 3, where we also describe in detail the transition between consensus and partial consensus. Finally, in Section 4, we conclude and discuss the implications of our work.

2. Simple model for co-evolution

Let us introduce a simplified version of CTD. In this version, the network is directed and all the nodes have two incoming links, i.e. each node is influenced by exactly two nodes, while their out-degree is initially Poisson distributed. Moreover, we will take $\phi = 1/2$, such that CTD now simplifies as follows. At each time step, a node i is selected at random. Let j and k be the two in-neighbours of i, namely the nodes at the extremities of its incoming links. If $s_i \neq s_j$ and $s_i \neq s_k$, node i switches its opinion, i.e., $s_i \to -s_i$. If the opinion of only one of its in-neighbours, say j, differs from s_i, i cuts its link from j and reconnects to a randomly chosen node, thereby maintaining its in-degree constant. If $s_i = s_j = s_k$, nothing happens. It is interesting to stress that this simplified version of CTD corresponds to the unanimity rule [20] when no rewiring is implemented. This model is well-known to exhibit a non-trivial relation between the initial and final densities of $+$ nodes, denoted by $n_{+;0}$ and $n_{+;\infty}$ respectively. We will show that the addition of the rewiring mechanism leads to a transition from a connected phase with consensus where all the nodes asymptotically belong to the same component, to a fragmented phase where two disconnected components of different opinions survive. The critical parameter of this transition is shown to be the initial density $n_{+;0}$ of $+$ nodes, i.e.,

$$n_{+;\infty} \begin{cases} = 0 \ (- \text{ consensus}) & \text{for } n_{+;0} < n_c, \\ \in]0, 1[\ (\text{fragmentation}) & \text{for } n_c < n_{+;0} < 1 - n_c, \\ = 1 \ (+ \text{ consensus}) & \text{for } n_{+;0} > 1 - n_c, \end{cases} \tag{1}$$

where n_c is the critical density.

3. Results

In order to analyze the system dynamics, let us follow the approach proposed in Ref. [20] and focus on the expected number $N_{s_0;s_1s_2}$ of configurations where a node in state s_0 receives its incoming links from a node in state s_1 and another node in state s_2. Let us denote by $\{s_0; s_1s_2\}$ such a triplet of nodes. By construction, s_i may be $+1$ or -1 and $\sum_{s_0s_1s_2} N_{s_0;s_1s_2} = N$. Moreover, the order of the links is not important and therefore $N_{s_0;s_1s_2} = N_{s_0;s_2s_1}$. By neglecting higher order correlations than those included in $N_{s_0;s_1s_2}$, it is possible to derive the set of equations

$$N_{+;++}(t+1) = N_{+;++} + \frac{1}{N}(N_{-;++} + n_+N_{+;+-} + \pi_{-\to+}N_{+;+-} - 2\pi_{+\to-}N_{+;++})$$
$$N_{+;--}(t+1) = N_{+;--} + \frac{1}{N}(-N_{+;--} + \pi_{+\to-}N_{+;+-} - 2\pi_{-\to+}N_{+;--})$$