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Pricing European options with a log Student's t-distribution: A Gosset formula

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ABSTRACT

The distributions of returns for stocks are not well described by a normal probability density function (pdf). Student's t-distributions, which have fat tails, are known to fit the distributions of the returns. We present pricing of European call or put options using a log Student's t-distribution, which we call a Gosset approach in honour of W.S. Gosset, the author behind the *nom de plume* Student. The approach that we present can be used to price European options using other distributions and yields the Black–Scholes formula for returns described by a normal pdf.

1. Introduction

In the fall of 2008, many investors and analysts witnessed multiple "once-in-a-lifetime" events in a single week, with disastrous consequences in many cases for the financial health of their portfolios. The once-in-a-lifetime designation of the events arises from calculation of the probabilities of events based on normal statistics. The returns were more than several sigma beyond the expected return, and by normal statistics, thought to be impossible.

The difficulty lies in the use of a normal pdf to describe returns. It is known that returns have "fat tails" [1,2], but for mathematical convenience and perhaps force of habit, a normal pdf is often applied. The celebrated Black–Scholes formula for pricing European options is based on several assumptions, one of which is that the returns are described by Brownian motion [3–5]. The underlying pdf for Brownian motion is a normal pdf.

In this paper we price European options using a log Student's t-distribution. We call these formulae of the prices of European options Gosset formulae, in honour of the author behind the *nom de plume* Student. We present evidence that stock returns are fit by a Student's t-distribution, in agreement with known results [6–9]. We then demonstrate the "fat tails" of the t-distribution. One of the difficulties with the fat tails is that one of the integrals, which is required to price an option, diverges. We present two similar approaches to handle successfully the divergence, and hence find a price for the option. These approaches can be used to price options using pdfs other than the t-distribution. However, we restrict our attention to the t-distribution and compare prices for options using the Gosset and Black–Scholes formulae. In general, the Gosset formula yields prices for options that are higher than the prices found from the Black–Scholes formula. This is not unexpected, as the Gosset formula gives greater weight to events in the tails.

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Heston [10] obtained a closed form solution for the price of options with stochastic volatility by using characteristic functions to solve partial differential equations (pde). Stochastic volatility leads to distributions with fat tails. The solution in the Heston approach is obtained by numerical computation of an integral. Numerical computation of an integral to find the price of an option is no different than finding the price of an option with the Black–Scholes formula, which requires numerical integration of the error function. Bouchaud and Sornette [11] obtained prices for options by a risk minimisation procedure that can be applied to more than just the stochastic process of uncorrelated Gaussian noise. The Black–Scholes formula for pricing options assumes Brownian motion, i.e., assumes uncorrelated Gaussian noise. Bouchaud and Sornette assumed that the average gain or loss should be zero and used this as their starting point to obtain prices by minimisation of risk. They considered the pricing of a European call option for a Lévy process and found that the cost of the option is infinite. They also recognized that truncation keeps the cost finite, but discounted the option price as being too sensitive to the level of truncation. Pinn [12] used the approach of Bouchaud and Sornette to study hedging and pricing of options using a Student's t-distribution. Pinn assumed, among other things, that the price of the stock was linearly and not exponentially related to the returns. This linear assumption keeps the integrals finite but also limits the applicability of the results to small changes. Lim et al. [13] studied the pricing of currency options using a generalized Student's t-distribution. The generalized t-distribution they employed was a t-distribution multiplied by a function of the form $\exp(-t^4)$. The exponential part of the generalized Student's t-distribution keeps the integrals needed to price the options finite. Lim et al. [13] assumed that the price of the option could be written as the expectation of the present value of the pay-off of the option. McCauley et al. [14] considered the pricing of options using distributions with fat tails. They used a Green's function approach to solve a pde and confirmed that the price of a call option is infinite with fat-tailed distributions. They considered using truncation to keep the integral finite, but felt that the option price would be too sensitive to the cut-off, meaning that essentially any option price could be predicted. Moriconi [15] studied hedging when the distribution for returns followed a Student's t-distribution multiplied by a wide Gaussian. The wide Gaussian effectively truncates the distribution and hence keeps the integrals that are required to price option finite. Moriconi solved a generalized Black–Scholes pde to obtain the price of a call.

In this paper we consider the pricing of European options using a Student's t-distribution. We find the price of the options using the arbitrage theorem and results from martingales. We argue that truncation or capping is physical and leads to realistic pricing. The equation that we obtain for the price of a European option is obtained without explicit solution of a pde [10,12,14,15] or functional minimisation [11]. The starting point for the pricing of the European options is the arbitrage theorem [16–18].

2. Arbitrage theorem

Consider an experiment that returns $r_i(e_j)$ for a bet of unity on outcome e_i of the N possible outcomes e_j of the experiment. In short, the arbitrage theorem states that either the mean return as calculated over the probability of the events e_j, $E\{r_i(e_j)\}$, is zero or there is a betting scheme that leads to a sure win. The betting scheme may allow for negative, zero, and positive bets. Puts and calls allow for negative, zero, and positive bets.

Let e_i be N disjoint events of which one and only one must occur. Let p_i be the probability of occurrence of event e_i and $\mathbf{c} = (c_1, c_2, \ldots, c_N)$ be a vector of bets on the N outcomes. The bets c_i may be zero, positive, or negative. The gain for a wager of c_i on the ith outcome given that event e_j occurred is $g_i(e_j) = c_i \times r_i(e_j)$. Note that in a typical bet, $r_i(e_j) = -1$ if $i \neq j$; the player forfeits the wager c_i if event i does not occur. The mean gain for a wager on the ith event is $\bar{g}_i = c_i \times \bar{r}_i$ and is the expectation over the e_j, which is obtained as $\sum_j p_j \times g_i(e_j) = c_i \times \sum_j p_j \times r_i(e_j)$.

Following the example of de Finetti [16], we recognize the $\bar{g}_i$ as N equations in N unknowns c_i. To ensure that no arbitrage exists, the average gain $\bar{g}_i = 0$. This is only possible for arbitrary c_i if $\bar{r}_i = 0$. If $\bar{r}_i \neq 0$, then the c_i can be selected to yield, on average, $\bar{g}_i > 0$. Thus to ensure no sure win, $\bar{r}_i = E\{r_i(e_j)\} = 0$. That this condition must hold for all i is a statement that there are no lucky numbers or values for assets. In the context of pricing an option on an asset, the events e_i could be the final price of the asset falling in a range.

Ross [18] shows how to price an option using the arbitrage theorem, assuming that the theorem can be generalized to handle functions rather than N discrete outcomes. If the cost of an option is C, then by the principle of no-arbitrage and the arbitrage theorem $C = E\{\rho_i\}$, where the return for a European call option is $\rho_i = (S_i - K, 0)^+$ and the return for a European put option is $\rho_i = (K - S_i, 0)^+$. The notation $(K - S, 0)^+$ means the maximum value of $(K - S)$ and 0, with K being the strike price and S the value of the asset.

Ross [18] obtains the Black–Scholes formula by assuming log normal statistics for the value of the asset and making some assumptions regarding the time value of the option. It is not necessary to solve the Black–Scholes partial differential equation to obtain the Black–Scholes formula for the price of a European option [18,19,10,14,15,12].

We accept as a starting point that the arbitrage theorem can be generalized from N discrete outcomes to hold for functions. We consider this as a reasonable starting point. The probability functions and values of assets can always be quantized to have N discrete ranges. We price European options assuming log Student's t-distributions, the arbitrage theorem, and results from martingale theory. The results from the martingale theory allow determination of the risk neutral price of the option.

First we demonstrate that the log Student's t-distribution is a reasonable distribution to use to price an option.