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Thermodynamics and dynamics of systems with long-range interactions

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ABSTRACT

We review simple aspects of the thermodynamic and dynamical properties of systems with long-range pairwise interactions (LRI), which decay as $1/r^{d+\sigma}$ at large distances r in d dimensions. Two broad classes of such systems are discussed. (i) Systems with a slow decay of the interactions, termed "strong" LRI, where the energy is super-extensive. These systems are characterized by unusual properties such as inequivalence of ensembles, negative specific heat, slow decay of correlations, anomalous diffusion and ergodicity breaking. (ii) Systems with faster decay of the interaction potential, where the energy is additive, thus resulting in less dramatic effects. These interactions affect the thermodynamic behavior of systems near phase transitions, where long-range correlations are naturally present. Long-range correlations are often present in systems driven out of equilibrium when the dynamics involves conserved quantities. Steady state properties of driven systems with local dynamics are considered within the framework outlined above.

1. Introduction

This paper provides a brief introduction to the thermodynamics and dynamics of systems with long-range interactions (LRI). In these systems, the interaction potential between the constituent particles decays slowly with distance, typically as a power law $\sim 1/r^{d+\sigma}$ at large separation $r \gg 1$, where d is the spatial dimension. The interaction potential may be isotropic or anisotropic (as in magnetic or electric dipolar systems). Long-range interacting systems may be broadly classified into two groups: those with $-d \leq \sigma \leq 0$, which are termed systems with "strong" LRI, and those with positive but not too large σ, which are termed systems with "weak" LRI. Systems with strong LRI show significant and pronounced dynamic and thermodynamic effects due to the slow decay of the interaction potential. In contrast, in systems with weak LRI, the potential decays relatively faster, resulting in less pronounced effects. For a recent review on long-range interacting systems, see Ref. [1].

Long-range interacting systems are rather common in nature, for example, self-gravitating systems ($\sigma = -2$) [2], non-neutral plasmas ($\sigma = -2$) [3], dipolar ferroelectrics and ferromagnets (anisotropic interactions with $\sigma = 0$) [4], two-dimensional geophysical vortices ($\sigma = -2$) [5], wave-particle interacting systems such as a free-electron laser [6], and many others.

Let us first consider systems with strong LRI. These systems are generically non-additive, resulting in many unusual properties, both thermal and dynamical, which are not exhibited by systems with weak LRI or with short-range interactions. For example, the entropy may turn out to be a non-concave function of energy, yielding negative specific heat within the microcanonical ensemble [7–14]. Since canonical specific heat is always positive, it follows that the two ensembles need not

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be equivalent. More generally, the inequivalence is manifested whenever a model exhibits a first-order transition within the canonical ensemble [15,16]. Non-additivity may also result in breaking of ergodicity, where the phase space is divided into domains. Local dynamics do not connect configurations in different domains, leading to finite gaps in macroscopic quantities such as the total magnetization in magnetic systems [17–23].

Studies of relaxation processes in models with strong LRI have shown that a thermodynamically unstable state relaxes to the stable equilibrium state unusually slowly over a timescale which diverges with the system size [5,17,24–30]. This may be contrasted with the relaxation process in systems with short-range interactions. Diverging timescales in systems with strong LRI result in long-lived quasistationary states. In the thermodynamic limit, these states do not relax to the equilibrium state, so that the system remains trapped in these states forever. These quasistationary states and their slow relaxation have been explained theoretically in the framework of kinetic theory [3,5,28–31]. Recent progress in the kinetic theory of systems with long-range interactions [31–33] has also uncovered algebraic relaxation and explained anomalous diffusion in and out of equilibrium.

It is worthwhile to point out that non-additivity may occur even in finite systems with short-range interactions in which surface and bulk energies are comparable. Negative specific heat in small systems (e.g., clusters of atoms) has been discussed in a number of studies [34–37].

Systems with weak LRI, for which $\sigma > 0$, are additive. Unless one is in the vicinity of a phase transition, their thermodynamic properties are similar to those with short-range interactions, e.g., the specific heat is non-negative, and the various statistical mechanical ensembles are equivalent. Near a phase transition, long-range correlations build up. These correlations affect the universality class of a system near a continuous phase transition, resulting in critical exponents which depend on the interaction parameter σ. Moreover, for these systems, the upper critical dimension $d_c(\sigma)$ above which the critical behavior becomes mean-field-like depends on σ and has a smaller value than in systems with short-range interactions for which $d_c = 4$, see Refs. [38,39]. A system with weak LRI may exhibit phase transitions in one dimension at a finite temperature, which are otherwise forbidden in a system with short-range interactions [40,41].

So far we have discussed systems in equilibrium. Long-range correlations may also build up in driven systems which reach a non-equilibrium steady state that violates detailed balance. Quite generally, such steady states in systems with conserving dynamics exhibit long-range correlations, even with local dynamics. One thus expects peculiarities in behavior of equilibrium systems with long-range interactions to also show up in steady states of non-equilibrium systems with conserving local interactions. An example of such a non-equilibrium system with long-range correlations is the so-called *ABC* model. In this model, three species of particles, *A*, *B* and *C*, move on a ring with local dynamical rules. At long times, the system reaches a nonequilibrium steady state in which the three species are spatially separated. The dynamics of this model lead to effective long-range interactions in the steady state [42,43].

The paper is laid out as follows. In Section 2, we discuss the thermodynamics and dynamics of systems with strong long-range interactions. This is followed by a discussion on upper critical dimension for systems with weak LRI in Section 3. The *ABC* model, exhibiting long-range correlations under out-of-equilibrium conditions is discussed in Section 4. The paper ends with conclusions.

2. Strong long-range interactions

2.1. Thermodynamics

Here we briefly discuss some general thermodynamic properties of systems with strong LRI. These systems are non-extensive and non-additive. For example, the energy of a particle interacting with a homogeneous distribution of particles in a volume V scales as $V^{-\sigma/d}$, so that the total energy scales superlinearly with the volume as $V^{1-\sigma/d}$, making it non-extensive, and hence, non-additive.

The most immediate consequence of non-additivity is that, unlike short-range systems, the entropy S is not necessarily a concave function of energy. This may be understood by referring to Fig. 1. The equilibrium state at a given energy within a microcanonical ensemble is obtained by maximizing the entropy at that energy. A short-range interacting system is unstable in the energy interval $E_1 < E < E_2$, since it can gain in entropy by phase separating into two subsystems with energies E_1 and E_2, keeping the total energy fixed. The energy and entropy densities are then given by the weighted average of the corresponding densities of the two coexisting subsystems. As a result, the physically realizable entropy curve in the unstable region is obtained by the common tangent line, resulting in an overall concave curve. However, in systems with strong LRI, due to non-additivity, the energy density of two coexisting subsystems is not given by the weighted average of the energy density of the two subsystems. Therefore, the non-concave curve of Fig. 1 could, in principle, represent a physically realizable stable system, with no occurrence of phase separation. This results in a microcanonical negative specific heat in the interval $E_1 < E < E_2$ [7–11,14]. Since the specific heat within the canonical ensemble is always positive, being given by the fluctuations about the mean of the system energy, this leads to inequivalence of ensembles, which is particularly manifested whenever a first-order transition with coexistence of two phases is found within the canonical ensemble [15,16].

Another feature related to non-additivity is that of a discontinuity in temperature at a first-order phase transition within a microcanonical ensemble, say, from a paramagnetic to a magnetically ordered phase. This may be understood by referring to Fig. 2(a), which shows the entropy $S(M, E)$ as a function of the magnetization M at an energy E close to the transition. It exhibits three local maxima, one at $M = 0$ and two other degenerate maxima at $M = \pm M_0$. As the