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Magnetization densities as replica parameters: The dilute ferromagnet

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ABSTRACT

In this paper we compute exactly the ground state energy and entropy of the dilute ferromagnetic Ising model. The two thermodynamic quantities are also computed when a magnetic field with random locations is present. The result is reached in the replica approach frame by a class of replica order parameters introduced by Monasson (1998) [5]. The strategy is first illustrated considering the SK model, for which we will show the complete equivalence with the standard replica approach. Then, we apply to the diluted ferromagnetic Ising model with a random located magnetic field, which is mapped into a Potts model.

1. Introduction

The theoretical modeling of statistical systems in many areas of physics makes use of random Hamiltonians. Assuming self-averaging, observable quantities may be evaluated by the logarithmic average of the partition function $\langle \log(Z) \rangle$ over the quenched random variables. In this way, the technically difficult task of computing Z for a given realization of the disorder is avoided. However, the mathematical operation of directly computing $\langle \log(Z) \rangle$ is also difficult, and can be done only using replica trick, which implies the computation of $\langle Z^n \rangle$.

Despite its highly successful application to the treatment of some disordered systems (the most celebrated is the SK model [1,2]), the replica trick encounters serious difficulties when applied to many other models. The main reason is the unbounded proliferation of replica order parameters, as for example the multi-overlaps in dilute spin glass [3,4].

In this paper we use a class of replica order parameters firstly introduced by Monasson [5], which, in principle can be used for all models. We preliminarily observe that the computation of $\langle Z^n \rangle$ implies the sum over all the realizations of the spin variables $\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^n)$, any of them may take 2^n values. Assume that $x(\sigma)N$ is the number of vectors σ_i which equal the given vector $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n)$, then, $\langle Z^n \rangle$ can be re-expressed in terms of a sum over the possible positive values of the 2^n magnetization densities $x(\sigma)$, with the constraint $\sum_\sigma x(\sigma) = 1$. In practice, this is equivalent to maximizing with respect to these order parameters.

The strategy proposed here will be illustrated, considering the SK model, in the next section, where we will also show the complete equivalence with the standard approach. Nevertheless, all models can be described in terms of the 2^n order parameters $x(\sigma)$. We will tackle the dilute ferromagnetic Ising model in Section 3 and in Section 4 we will compute exactly the ground state energy and entropy. The results coincide with those found in Ref. [6] where the approach is not based on replicas. In Section 5 we extend the scope by considering the same model in the presence of a magnetic field with random locations, and, finally, in Section 6 we are able to compute the ground state energy and entropy also in this case. Conclusions and outlook are in the final section.

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2. SK spin glass

In this section we try to illustrate our approach considering the SK model. We do not have new results concerning SK, but we just show the complete equivalence with the standard replica approach.

The partition function is

$$Z = \sum_{\#} \exp\left(\frac{\beta}{\sqrt{N}} \sum_{i>j} J_{ij}\sigma_i\sigma_j \right) \tag{1}$$

where the sum $\sum_{\#}$ goes over the 2^N realizations of the σ_i, the sum $\sum_{i>j}$ goes over all $N(N-1)/2$ pairs ij and the J_{ij} are independent random variables with 0 mean and variance 1. Then, by replica approach, neglecting terms which vanish in the thermodynamic limit, we have

$$\langle Z^n \rangle = \sum_{\#} \exp\left(\frac{\beta^2}{4N} \sum_{i,j} \left(\sum_{\alpha} \sigma_i^{\alpha}\sigma_j^{\alpha} \right)^2 \right) \tag{2}$$

where the sum $\sum_{\#}$ goes over the 2^{nN} realizations of the σ_i^{α} and the sum $\sum_{i,j}$ goes over all N^2 possible values of i and j.

Assume that $Nx(\sigma)$ is the number of vectors $(\sigma_i^1, \sigma_i^2, \dots, \sigma_i^n)$ which equal the given vector $\sigma = (\sigma^1, \sigma^2, \dots, \sigma^n)$, then, according to Ref. [5], we can write

$$\sum_{i,j} \left(\sum_{\alpha} \sigma_i^{\alpha}\sigma_j^{\alpha} \right)^2 = N^2 \sum_{\sigma,\tau} x(\sigma)x(\tau)(\sigma\cdot\tau)^2 \tag{3}$$

where $\sum_{\sigma,\tau}$ goes over the 2^{2n} possible values of the variables σ and τ and where $\sigma\cdot\tau$ is the scalar product $\sigma\cdot\tau = \sum_{\alpha} \sigma^{\alpha}\tau^{\alpha}$. Then, observe that the number of realizations corresponding to a given value of the 2^n magnetization densities $x(\sigma)$ is

$$\exp\left(-N \sum_{\sigma} x(\sigma)\log(x(\sigma)) \right) \tag{4}$$

where $\sum_{\sigma}$ is the sum over the 2^n possible values of the variable σ. Indeed, in the above expression we neglected terms which are in-influent in the thermodynamic limit.

We can now define Φ_n as the large N limit of $\frac{1}{N}\log\langle Z^n\rangle$, then

$$\Phi_n = \max_x \left[\frac{\beta^2}{4} \sum_{\sigma,\tau} x(\sigma)x(\tau)(\sigma\cdot\tau)^2 - \sum_{\sigma} x(\sigma)\log(x(\sigma)) \right]. \tag{5}$$

The maximum is taken over the possible values of the 2^n order parameters $x(\sigma)$ provided that $\sum_{\sigma} x(\sigma) = 1$. The constraint can be accounted by adding the Lagrangian multiplier $\lambda(\sum_{\sigma} x(\sigma) - 1)$ to expression (5), then the maximum is given by

$$\frac{\beta^2}{2} \sum_{\tau} x(\tau)(\sigma\cdot\tau)^2 - \log(x(\sigma)) - 1 = \lambda \tag{6}$$

where λ has to be chosen in order to have $\sum_{\sigma} x(\sigma) = 1$. From this equation we get that the maximum is realized for the set of the 2^n order parameters $x(\sigma)$ which satisfy

$$x(\sigma) = \frac{1}{A} \exp\left(\frac{\beta^2}{2} \sum_{\tau} x(\tau)(\sigma\cdot\tau)^2 \right) \tag{7}$$

where the sum $\sum_{\tau}$ goes over the 2^n possible values of the variable τ and where A is

$$A = \sum_{\sigma} \exp\left(\frac{\beta^2}{2} \sum_{\tau} x(\tau)(\sigma\cdot\tau)^2 \right) \tag{8}$$

and where $\sum_{\sigma}$ is the sum over the 2^n possible values of the variable σ.

The explicit solution could be found, in principle, by a proper choice of the parametrization of the $x(\sigma)$, but it can be easily seen that this solution coincides with the standard solution of the SK model. In fact, if one defines the symmetric matrix $q_{\alpha\beta}$ as

$$q_{\alpha\beta} = \sum_{\tau} x(\tau)\tau^{\alpha}\tau^{\beta} \tag{9}$$