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On invariant 2×2 β-ensembles of random matrices

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ABSTRACT

We introduce and solve exactly a family of invariant 2×2 random matrices, depending on one parameter η, and we show that rotational invariance and real Dyson index β are not incompatible properties. The probability density for the entries contains a weight function and a multiple trace–trace interaction term, which corresponds to the representation of the Vandermonde-squared coupling on the basis of power sums. As a result, the effective Dyson index β_{eff} of the ensemble can take any real value in an interval. Two weight functions (Gaussian and non-Gaussian) are explored in detail and the connections with β-ensembles of Dumitriu–Edelman and the so-called Poisson–Wigner crossover for the level spacing are respectively highlighted. A curious spectral twinning between ensembles of different symmetry classes is unveiled: as a consequence, the identification between symmetry group (orthogonal, unitary or symplectic) and the exponent of the Vandermonde ($\beta = 1, 2, 4$) is shown to be potentially deceptive. The proposed technical tool more generically allows for designing actual matrix models which (i) are rotationally invariant; (ii) have a real Dyson index β_{eff}; (iii) have a pre-assigned confining potential or alternatively level-spacing profile. The analytical results have been checked through numerical simulations with an excellent agreement. Eventually, we discuss possible generalizations and further directions of research.

1. Introduction

Ensembles of matrices with random elements have been widely studied since the pioneering works of Wigner [1] and Dyson [2] on the 'threefold way'. A first, gross classification of random matrix (RM) models can take into account (i) whether the size N of the matrices in the ensemble is finite or the limit $N \to \infty$ is taken and (ii) whether the probability distribution of the entries remains invariant after a rotation in the matrix space.

The requirement of rotational invariance implies that the joint probability density (jpd) of the eigenvalues can be written as:

$$P(\lambda_1, \dots, \lambda_N) \propto \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{N} V(\lambda_i)} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta} \tag{1}$$

where $V(x)$ is a confining potential (x^2 for Gaussian ensembles) and the interaction term between eigenvalues is the well-known Vandermonde determinant raised to the power β. The Dyson index β can classically take *only* the values 1, 2, 4 according to the number of variables needed to specify a single entry (1 for real, 2 for complex and 4 for quaternion numbers). This β-index in turn identifies the symmetry group of the ensemble (Orthogonal, Unitary and Symplectic respectively).

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Thanks to the works of Mehta [3] and many others, very powerful analytical tools are available to deal with invariant ensembles, both for finite N and as $N \to \infty$, the latter limit being usually the most interesting for RM theorists. However, it was very soon realized that matrices with the smallest size $N = 2$ can equally well provide deep insights and trigger new ideas, the most successful one being the celebrated Wigner's surmise [3] which gives an excellent approximation for the level spacing of bigger matrices. The study of 2×2 random matrices has since been strongly developed and it remains an active area of research in mathematical physics [4–17].

The purpose of the present paper is to introduce and solve exactly a family of 2×2 random matrices depending on one parameter η. This ensemble will have rotational invariance *but* a real effective Dyson index β_{eff} in an interval. Although it is commonly assumed that the two properties:

- rotational invariance;
- real Dyson index.

are essentially incompatible, since the Dyson index of an invariant ensemble is strictly constrained to the values 1, 2 or 4 as described above, we will show how to construct explicitly a counterexample in Section 2 introducing suitable correlations among the matrix entries. The motivation for this study stems from two apparently unrelated issues, namely the Dumitriu–Edelman β-ensembles [18] and the so-called Poisson–Wigner crossover for the level spacing [19]. In order to make the paper self-contained, we give a brief introduction to both of them highlighting also the two main tasks we tackle in this paper. In Section 1.3, we provide the plan of the article.

1.1. β-ensembles of Dumitriu–Edelman

Consider the jpd (1). Does there exist a non-trivial matrix model having (1) as its jpd of eigenvalues for *any* $\beta > 0$? Very recently, Dumitriu and Edelman were able to answer this question affirmatively [18]. They introduced two ensembles of tridiagonal $N \times N$ matrices with independent entries, whose jpd of eigenvalues is exactly given by (1) for general $\beta > 0$ [18]. These ensembles have been called β-Hermite and β-Laguerre, according to the classical weight their jpd contains. This result is essential for an efficient numerical sampling of random matrices [20] and has triggered a significant amount of further research [21–25].

Note that the β-ensembles, having independent non-Gaussian entries are obviously *non-invariant*. Thus, the first novel task we tackle in this paper (Section 3) is the following:

Task 1. *Design and solve exactly a* (2×2) *ensemble with:*

- *rotational invariance;*
- *running* $\beta_{\text{eff}} \geq 0$[1]*;*
- *assigned classical potential (in particular, Gaussian-Hermite).*

In fact, an invariant matrix model displaying a running Dyson index would be of great interest: tuning the strength of the correlations between the eigenvalues in (1) has significant importance for systems which, although endowed with an intrinsic invariance, are subjected to a weak non-invariant perturbation (see e.g. Ref. [26]) and may also have important implications for lattice gas theory [27]. Furthermore, it is a long-standing observation that nuclear systems with two-body interactions display an average density of states whose profile is much closer to a Gaussian distribution [28,29] than to a semicircle. Hence, a RM approach with the appropriate symmetries clearly requires much weaker, and possibly suppressed altogether, correlations among the energy levels than those arising from (1) with integer and fixed β. In this respect, the limit $\beta_{\text{eff}} \to 0$ of our model is particularly appealing (see Section 3).

1.2. Poisson–Wigner crossover

Another interesting transitional regime in quantum chaos theory, namely the so-called Poisson–Wigner crossover for level spacings, has attracted much attention in the past twenty years [19]. In terms of the dimensionless nearest-neighbor spacing s, the Poisson and Wigner distributions are given by:

$$P_{\text{POI}}(s) = \exp(-s) \tag{2}$$

$$P_{\text{WIG}}(s) = \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right) \tag{3}$$

and correspond to the limiting cases of classical dynamics, namely purely regular and completely chaotic. Intermediate regimes between those two extremes have been intensely investigated (see Ref. [30] for a review), and interpolating phenomenological formulas have been proposed, the most famous being the Brody [31] and Berry–Robnik [32] distributions. The quest for a deeper understanding of such a crossover has motivated many proposals of parametrical random matrix models whose level-spacing distribution interpolates between (2) and (3) [33–35,4–6]. Normally, the requirement of

[1] Comments on the case $\beta_{\text{eff}} \equiv 0$ are given in Section 3.