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Deterministic versus stochastic aspects of superexponential population growth models

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HIGHLIGHTS

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- Lamperti transforms.
- Lamperti processes.
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ABSTRACT

Deterministic population growth models with power-law rates can exhibit a large variety of growth behaviors, ranging from algebraic, exponential to hyperexponential (finite time explosion). In this setup, selfsimilarity considerations play a key role, together with two time substitutions. Two stochastic versions of such models are investigated, showing a much richer variety of behaviors. One is the Lamperti construction of selfsimilar positive stochastic processes based on the exponentiation of spectrally positive processes, followed by an appropriate time change. The other one is based on stable continuous-state branching processes, given by another Lamperti time substitution applied to stable spectrally positive processes.

1. Introduction

Deterministic population growth models (1) with power-law rates μx^{γ}, $\mu > 0$, can exhibit a large variety of behaviors, ranging from algebraic ($\gamma < 1$), exponential ($\gamma = 1$) to hyperexponential (finite time explosion if $\gamma > 1$) growth for the size (or mass) $x(t)$ of some population at time $t \geq 0$. The exponential (Malthusian) growth regime with $\gamma = 1$ discriminates between the two other ones and the transition at $\gamma = 1$ is quite sharp. In this setup, selfsimilarity considerations (with Hurst index $\alpha = 1/(1-\gamma)$) play a key role, together with two time substitutions. Log-selfsimilarity considerations can also be introduced while exponentiating the latter model for $x(t)$. In this setup, the discriminating process grows at double (or superexponential) speed. This discriminating process separates two log-self-similar processes, one growing at exp-algebraic rate and the other one blowing-up in finite time.

In this manuscript, two stochastic versions of such population growth models with similar flavor are investigated, showing a much richer variety of behaviors. One is the Lamperti construction of selfsimilar positive stochastic processes

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based on the exponentiation of spectrally positive processes, followed by an appropriate time change. As an example, the Lamperti diffusion process (22) is studied in some detail, including the noncritical cases. For the critical case with $\mu = 0$ for instance, we show that the transition $\gamma < 1$ to $\gamma > 1$ is rather smooth: indeed, if $\gamma < 1$, state ∞ is a natural inaccessible boundary whereas state 0 is exit (or absorbing) and reached eventually in finite time. The population dies out (extinction) fast. If $\gamma = 1$ (when the discriminating critical process is geometric Brownian motion), state ∞ is an entrance state and state 0 a natural inaccessible boundary. State 0 (extinction) is reached eventually but now not in finite time. If $\gamma > 1$, state 0 is a natural inaccessible boundary whereas state ∞ is an entrance state. The process drifts to ∞ (explosion) but not in finite time. Situations for which there is a finite time explosion can occur but only in noncritical cases when $\gamma > 1$ and μ exceeds some positive threshold. In all cases, depending on $\gamma < 1$ ($\gamma > 1$), such processes are stochastically selfsimilar with Hurst index $\alpha > 0$ ($\alpha < 0$).

The other one is based on continuous-state branching processes (CSBPs) $\mathrm{x}(t)$, as given by another Lamperti time substitution of spectrally positive processes: in this respect, the a-stable Lamperti CSBP (with $a \in (1, 2)$) and the one-sided a-stable CSBP (with $a \in (0, 1)$) are investigated in some detail. Both noncritical and critical cases are considered. The critical version of these models is shown to exhibit self-similarity properties: the obtained Hurst indices are $\alpha = 1/(a-1)$ with range $\alpha > 1$ and $\alpha < -1$, respectively. Taking $a \to 1^{\pm}$ yields in the first place the deterministic Malthusian growth model: $\mathrm{x}(t) = \mathrm{x}e^{(\mu \pm \kappa)t}$. This Malthusian regime separates a situation for which $\mathbf{E}(\mathrm{x}(t) \mid \mathrm{x}(t) > 0) \propto t^{\alpha}$ has superlinear algebraic growth rate (for the a-Lamperti model) and a situation for which $\mathrm{x}(t)$ is not regular as it blows up for all time $t > 0$ (for the one-sided a-stable model). The Malthus model is the discriminating critical process of such CSBP population growth models and the situation looks quite similar to the deterministic setup, although much more complex. The transition at $a = 1$ is sharp. While considering a different limiting process as $a \to 1^{\pm}$, we obtain the Neveu CSBP model which grows a.s. at double superexponential speed. The critical version of this process is no longer self-similar. It plays the role of the superexponential discriminating deterministic model separating two log-self-similar models: the exp-algebraic and the blowing-up regimes, respectively.

2. Deterministic population growth models

2.1. A class of self-similar growth models

Let $x(t) \geq 0$ denote the size (mass) of some population at time $t \geq 0$, with initially $x := x(0) \geq 0$. With $\mu, \gamma > 0$, consider the growth dynamics

$$\dot{x}(t) = \mu x(t)^{\gamma}, \qquad x(0) = x, \tag{1}$$

for some velocity field $v(x) = \mu x^{\gamma}$. Integrating when $\gamma \neq 1$ (the nonlinear case), we get formally

$$x(t) = \left(x^{1-\gamma} + \mu(1-\gamma)t\right)^{1/(1-\gamma)}. \tag{2}$$

Three cases arise:

• $0 < \gamma < 1$: then $x \geq 0$ makes sense and in view of $1/(1-\gamma) > 1$, the growth of $x(t)$ is algebraic at rate larger than 1. We note that $x(t, x) := x(t)$ with $x(0) = x$ obeys the selfsimilarity property: for all $\lambda > 0$, $t \geq 0$ and $x \geq 0$, $x(\lambda t, \lambda^{\alpha} x) = \lambda^{\alpha} x(t, x)$, with $\alpha := 1/(1-\gamma) > 1$, the Hurst exponent. When $x = 0$, the dynamics has two solutions, one $x(t, 0) \equiv 0$ for $t \geq 0$ and the other $x(t, 0) = (\mu(1-\gamma)t)^{1/(1-\gamma)}$ because the velocity field v in (1) with $v(0) = 0$, is not Lipschitz as x gets close to 0, having an infinite derivative. The solution $x(t, 0) = (\mu(1-\gamma)t)^{1/(1-\gamma)}$ with $x = 0$ reflects some spontaneous generation phenomenon: following this path, the mass at time $t > 0$ is not 0, although initially it was.

• $\gamma > 1$: then $x > 0$ only makes sense and explosion or blow-up of $x(t)$ occurs in finite time $t_{\mathrm{exp}} = x^{1-\gamma}/[\mu(\gamma-1)]$. Up to the explosion time t_{exp}, $x(t)$ is selfsimilar with Hurst exponent $\alpha = 1/(1-\gamma) < 0$. Whenever $x(t)$ blows up in finite time, following [1], we shall speak of a hyperexponential growth regime. This model was shown to be meaningful as a world population growth model over the last two millenaries, [1]. There is also some recent empirical interest into models with similar behavior in Refs. [2–4]. The finite-time explosion feature, the related interpretation problems and the previous works about this interpretation have been emphasized in Ref. [5], where the author considers the technological advance of a given market. More technically, necessary and sufficient conditions for the existence of such a blowing up regime involving the asymptotic form of the local series representation for the general solutions around the singularities are given in Ref. [6].

• $\gamma = 1$: this is a simple special case not treated in (2), strictly speaking. However, expanding the solution (2) in the leading powers of $1-\gamma$ yields consistently:

$$\begin{aligned} x(t) &= e^{\log\left(x^{1-\gamma}+\mu(1-\gamma)t\right)/(1-\gamma)} \\ &= e^{\log[x^{1-\gamma}\left(1+\mu x^{\gamma-1}(1-\gamma)t\right)]/(1-\gamma)} \sim x e^{(1/(1-\gamma))\mu x^{\gamma-1}(1-\gamma)t} \sim x e^{\mu t}. \end{aligned} \tag{3}$$

Here $x \geq 0$ makes sense for (1) with $x(t) = x e^{\mu t}$ for $t \geq 0$ if $x \geq 0$. This is the simple Malthus growth model. The Malthus regime with $\gamma = 1$ will be called "discriminating" for (1), in the sense that it separates a slow algebraic growth regime ($\gamma < 1$) and a blowing-up regime ($\gamma > 1$).