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Transient dynamics of sparsely connected Hopfield neural networks with arbitrary degree distributions

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Abstract

Using probabilistic approach, the transient dynamics of sparsely connected Hopfield neural networks is studied for arbitrary degree distributions. A recursive scheme is developed to determine the time evolution of overlap parameters. As illustrative examples, the explicit calculations of dynamics for networks with binomial, power-law, and uniform degree distribution are performed. The results are good agreement with the extensive numerical simulations. It indicates that with the same average degree, there is a gradual improvement of network performance with increasing sharpness of its degree distribution, and the most efficient degree distribution for global storage of patterns is the delta function.

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As a tractable toy model of associative memories and can also be viewed as an extension of the Ising model, Hopfield neural networks [1] received lots of attention in the recent two decades. Equilibrium properties of fully connected Hopfield neural networks have been well studied using spin-glass theory, especially the replica method [2,3]. Their dynamics is also studied using the generating functional method [4] and signal-to-noise analysis [5–7].

Given the huge number of neurons, there is only a small number of interconnections in the human brain cortex ($\sim 10^{11}$ neurons and $\sim 10^{14}$ synapses). In order to simulate a biological genuine model rather than the fully connected networks, various random diluted models were studied, including extremely diluted model [8,9], finite diluted model [10,11], and finite connection model [12,13]. But neural connectivity is suggested to be far more complex than a fully random graph, e.g. the networks of c.elegans and cat's cortical neural were reported to be small-world and scale-free, respectively [14,15]. To go one step closer to a more biologically realistic model, many numerical studies are carried out, focusing on how the topology, the degree distribution, and clustering coefficient of a network topology affect the computational performance of the Hopfield model [16–19]. With the same average connection, a random network was reported to be more efficient for storage and retrieval of patterns than either a small-world network or regular network [17]. Torres et al. reported that the capacity of storage is higher for a neural network with

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scale-free topology than for highly random diluted Hopfield networks [18]. However, to our best knowledge, there are no theoretical results of either dynamics or statics yet.

The goal of this paper is to analytically study the dynamics of the Hopfield model for a sparsely connected topology whose degree distribution is not restricted to a specific distribution (e.g. Poisson) but can take arbitrary forms. Another question investigated in this paper is how the degree distribution of connection topology influences the network performance, especially whether there exists an optimal degree distribution given a fixed number of nodes and connections.

Let us consider a system of N spins or neurons, the state of the spins takes $s_i(t) = \pm 1$ and updates synchronously with the following probability,

$$\mathrm{Prob}[s_i(t+1)|h_i(t)] = \frac{\mathrm{e}^{\beta s_i(t+1)h_i(t)}}{2\cosh(\beta h_i(t))}, \tag{1}$$

where β is the inverse temperature and the local field of neuron i is defined by

$$h_i(t) = \sum_{j=1}^{N} J_{ij} s_j(t). \tag{2}$$

We store $q = \alpha N$ random patterns $\xi^\mu = \left(\xi_1^\mu, \ldots, \xi_N^\mu\right)$ in networks, where α is called the loading ratio. The couplings are given by the Hebb rule,

$$J_{ij} = \frac{C_{ij}}{N} \sum_{\mu=1}^{q} \xi_i^\mu \xi_j^\mu, \tag{3}$$

where C_{ij} is the adjacency matrix ($C_{ij} = 1$ if j is connected to i, $C_{ij} = 0$ otherwise). In contrast to spin glasses or many other physical systems, the interactions between biological neurons are not symmetric: neuron i may influence neuron j even if neuron j has no influence on neuron i. So in our model, C_{ij} and C_{ji} are chosen independently. Degree of spin i, $k_i = \sum_{j=1}^{N} C_{ij}$, denotes the number of spins that are connected to i. We consider the case that neurons are sparsely connected, it means that $N \to \infty$, $k_i \to \infty$ but $k_i/N \to 0$. For example, we can take $k_i = O(\ln N)$. And in this paper, the degrees of neurons are set as an arbitrary distribution $p\,(k_i = k)$.

We use $g\,(\cdot)$ to express the transfer function,

$$s_i(t+1) = g\,(h_i(t))\,. \tag{4}$$

Without loss of generality, let us consider the case to retrieve ξ^1. We define $m(t)$ as the overlap parameter between network state $s\,(t)$ and the first pattern ξ^1 as

$$m(t) = \frac{1}{N} \sum_{i=1}^{N} \xi_i^1 s_i(t). \tag{5}$$

Then the local field at time t can be represented by

$$h_i(t) = \frac{1}{N} \sum_{j \neq i}^{N} C_{ij} \xi_i^1 \xi_j^1 s_j(t) + \frac{1}{N} \sum_{\mu \neq 1}^{q} \sum_{j \neq i}^{N} C_{ij} \xi_i^\mu \xi_j^\mu s_j(t), \tag{6}$$

where the first term is the signal from ξ^1 and the second one is crosstalk noise from other patterns. Our aim is to determine the form of the local field in the thermodynamic limit $N \to \infty$. We apply the law of large numbers to the signal term and find that it converges to $\xi_i^1 \frac{k_i}{N} m(t)$ in the thermodynamic limit. To show this point intuitively, we can simply replace the signal term by its average,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j \neq i} C_{ij} \xi_i^1 \xi_j^1 s_j(t) = \xi_i^1 \left\langle C_{ij} \xi_j^1 s_j(t) \right\rangle. \tag{7}$$