Available online at www.sciencedirect.com
ScienceDirect
Physica A 387 (2008) 1056–1062
PHYSICA A
www.elsevier.com/locate/physa

Performance comparison between classical and quantum control for a simple quantum system[☆]

Zairong Xi[a,*], Guangsheng Jin[b]

[a] *Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100080, China*
[b] *Institute of Physics, Chinese Academy of Sciences, Beijing, 100080, China*

Received 23 June 2007; received in revised form 8 August 2007
Available online 12 October 2007

Abstract

Brańczyk et al. pointed out that the quantum control scheme is superior to the classical control scheme for a simple quantum system using simulation [A.M. Brańczyk, P.E.M.F. Mendonca, A. Gilchrist, A.C. Doherty, S.D. Barlett, Quantum control theory of a single qubit, Physical Review A 75 (2007) 012329 or arXiv e-print quant-ph/0608037]. Here we rigorously prove the result. Furthermore we will show that any quantum operation does not universally "correct" the dephasing noise.

Keywords: Quantum control; Quantum information; Quantum channel; Quantum noise; Classical control

1. Introduction

Recently, Brańczyk et al. considered the classical and quantum control for a simple quantum system in Ref. [2]. They considered the following operational task: a qubit prepared in one of the two non-orthogonal states $|\psi_1\rangle$ or $|\psi_2\rangle$ (with overlap $\langle\psi_1|\psi_2\rangle = \cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$) is transmitted along a quantum channel with the dephasing noise, i.e., with probability p a Pauli operator Z is applied to the system, and with probability $1 - p$ the system is unaltered, where $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$ and $\{|0\rangle, |1\rangle\}$ be a basis for the qubit Hilbert space. The noise is thus described by a quantum operation, i.e., a completely-positive trace-preserving (CPTP) map $\mathcal{E}_p$, that acts on a single-qubit density matrix ρ as

$$\mathcal{E}_p(\rho) = pZ\rho Z + (1 - p)\rho. \tag{1}$$

Suppose the noisy channel is to be fully characterized, meaning that p is known and without loss of generality in the range $0 \le p \le 0.5$. Two initial states to be oriented are chosen as

[☆] This research is supported by the National Natural Science Foundation of China (No. 60774099, No. 60221301) and by the Chinese Academy of Sciences (KJCX3-SYW-S01).

* Corresponding author.
E-mail address: zrxi@iss.ac.cn (Z. Xi).

0378-4371/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2007.10.019

$$\begin{cases} |\psi_1\rangle = \cos\dfrac{\theta}{2}|+\rangle + \sin\dfrac{\theta}{2}|-\rangle, \\ |\psi_2\rangle = \cos\dfrac{\theta}{2}|+\rangle - \sin\dfrac{\theta}{2}|-\rangle, \end{cases} \tag{2}$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, in which their distinguishability is maintained under the dephasing noise by their trace distance. It is easy to see $Z|+\rangle = |-\rangle$, $Z|-\rangle = |+\rangle$.

To quantify the performance of control scheme, the average fidelity is used to compare the noiseless input states $|\psi_i\rangle$ with the corrected output states ρ_i. Assuming an equal probability for sending either state $|\psi_1\rangle$ or $|\psi_2\rangle$, the performance index is

$$\begin{aligned} F_{\mathcal{C}} &= \frac{1}{2}F(|\psi_1\rangle, \rho_1) + \frac{1}{2}F(|\psi_2\rangle, \rho_2) \\ &= \frac{1}{2}\langle\psi_1|\rho_1|\psi_1\rangle + \frac{1}{2}\langle\psi_2|\rho_2|\psi_2\rangle, \end{aligned} \tag{3}$$

where the fidelity between a pure state $|\psi\rangle$ and a mixed state ρ is defined as $F(|\psi\rangle, \rho) \equiv \langle\psi|\rho|\psi\rangle$. The fidelity F ranges from 0 to 1 and is a measure of how much the two states overlap each other (a fidelity of 0 means the states are orthogonal, whereas a fidelity of 1 means the states are identical). It has the following simple operational meaning when the input state is pure: the fidelity $F(|\psi_i\rangle, \rho_i)$ is the probability that the state ρ_i will yield outcome $|\psi_i\rangle$ from the projective measurement $\{|\psi_i\rangle\langle\psi_i|, |\psi_i^\perp\rangle\langle\psi_i^\perp|\}$. They considered two kinds of control schemes, i.e., classical and quantum schemes, to "correct" the system without knowing which state was transmitted, i.e., undo the effect of the noise, through the use of a control scheme based on measurement and feedback.

Classical control

For the considered case, the optimal measurement given by Helstrom [3] is a projective measurement onto the basis $\{|0\rangle, |1\rangle\}$ in terms of maximizing the average probability of a success, which successfully discriminates the states $|\psi_1\rangle$ and $|\psi_2\rangle$ with probability $P_{\text{Helstrom}} = \frac{1}{2}(1 + \sin\theta)$. Then an entanglement-breaking trace-preserving (EBTP) scheme is used [4,8], which arises because the output system is unentangled with any other system, regardless of its input state. Through this scheme the optimal performance was proved to be

$$F_{DR2} = \frac{1}{2} + \frac{1}{2}\sqrt{\sin^4\theta + \cos^2\theta} \tag{4}$$

when preparing states

$$|\Psi^{\pm}\rangle = \sqrt{\frac{1}{2} \pm \frac{\sin^2\theta}{2\gamma}}|0\rangle + \sqrt{\frac{1}{2} \mp \frac{\sin^2\theta}{2\gamma}}|1\rangle \tag{5}$$

corresponds to measurement results 0, respectively, 1 where $\gamma = \sqrt{\cos^2\theta + \sin^4\theta}$.

Do nothing

Doing nothing to correct the states is also a classical control strategy. The average fidelity of this scheme is given by

$$F_N = 1 - p\cos^2\theta. \tag{6}$$

Random preparation

Another control strategy would be to prepare the states randomly with probability $\frac{1}{2}$, i.e., $\frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$, which is not considered in Ref. [2]. Although trivial, this strategy is of interest for comparison with other schemes. This scheme also is not described by an EBTP map, but we will nonetheless refer to it as "classical".

The average fidelity of this scheme is given by

$$F_E = \frac{1}{2}(1 + \cos^2\theta). \tag{7}$$