Physica A 390 (2011) 419–426

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Asymptotic behavior for a version of directed percolation on a square lattice

Lung-Chi Chen

Department of Mathematics, Fu-Jen Catholic University, Taiwan

ARTICLE INFO

Article history:
Received 28 July 2010
Received in revised form 10 September 2010
Available online 19 October 2010

Keywords:
Domany–Kinzel model
Directed percolation
Compact directed percolation
Asymptotic behavior
Critical behavior
Susceptibility
Exact solutions
Berry–Esseen theorem

ABSTRACT

We consider a version of directed bond percolation on a square lattice whose vertical edges are directed upward with probabilities p_v and horizontal edges are directed rightward with probabilities p_h and 1 in alternate rows. Let $\tau(M, N)$ be the probability that there is a connected directed path of occupied edges from $(0, 0)$ to (M, N). For each $p_h \in [0, 1]$, $p_v = (0, 1)$ and aspect ratio $\alpha = M/N$ fixed, it was established (Chen and Wu, 2006) [9] that there is an $\alpha_c = [1 - p_v^2 - p_h(1 - p_v)^2]/2p_v^2$ such that, as $N \to \infty$, $\tau(M, N)$ is 1, 0, and 1/2 for $\alpha > \alpha_c, \alpha < \alpha_c$, and $\alpha = \alpha_c$, respectively. In particular, for $p_h = 0$ or 1, the model reduces to the Domany–Kinzel model (Domany and Kinzel, 1981 [7]). In this article, we investigate the rate of convergence of $\tau(M, N)$ and the asymptotic behavior of $\tau(M_n^-, N)$ and $\tau(M_n^+, N)$, where $M_n^-/N \uparrow \alpha_c$ and $M_n^+/N \downarrow \alpha_c$ as $N \uparrow \infty$. Moreover, we obtain a susceptibility on the rectangular net $\{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 0 \le m \le M \text{ and } 0 \le n \le N\}$. The proof is based on the Berry–Esseen theorem.

1. Introduction

Directed percolation (oriented percolation) can be thought of simply as a percolation process on a directed lattice in which connections are allowed only in a preferred direction. It was first studied by Broadbent and Hammersley in 1957 [1] and it has remained to this day as one of the most outstanding interesting problems in probability and statistical mechanics. Furthermore, directed percolation is closely related to the Reggeon field theory in high-energy physics and the Markov processes with branching, recombination, and absorption that occur in chemistry and biology [2,3], etc. Various properties, results, and conjectures of directed percolation can be found in [4,5] and the references therein. However, very little is known in the way of exact solutions for the directed percolation problem.

We say that the vertex (m, n) is percolating if there is a connected directed path of occupied edges from $(0, 0)$ to (m, n). Compact directed percolation [6] is a version of the universality of directed percolation class. It is defined on a square by the condition transition probabilities as follows: $P((x, y)$ is percolating: $(x-1, y), (x, y-1)$ are not percolating$) = 0$, $P((x, y)$ is percolating: $(x-1, y)$ is percolating and $(x, y-1)$ is not percolating$) = p_1$, $P((x, y)$ is percolating: $(x-1, y)$ is not percolating and $(x, y-1)$ is percolating$) = p_2$ and $P((x, y)$ is percolating: $(x-1, y), (x, y-1)$ are percolating$) = 1$ for any $p_1, p_2 \in [0, 1]$ and $(x, y) \neq (0, 0)$. Hence, the system has two absorbing states, namely, the empty and the fully occupied lattice.

Domany and Kinzel [7] defined a solvable version of compact directed percolation on a square lattice in 1981, as follows. For $p \in (0, 1)$ fixed, each nearest-neighbour vertical bond is directed upward with occupation probability p (independently of the other bonds) and each nearest-neighbour horizontal bond is directed rightward with occupation probability 1. Furthermore, it is known that the boundary of the Domany–Kinzel model has the same distribution as the one-dimensional last-passage percolation model [8]. In this article, we consider a version of directed percolation on a square lattice whose

E-mail address: lcchen@math.fju.edu.tw.

0378-4371/$ – see front matter
doi:10.1016/j.physa.2010.09.039

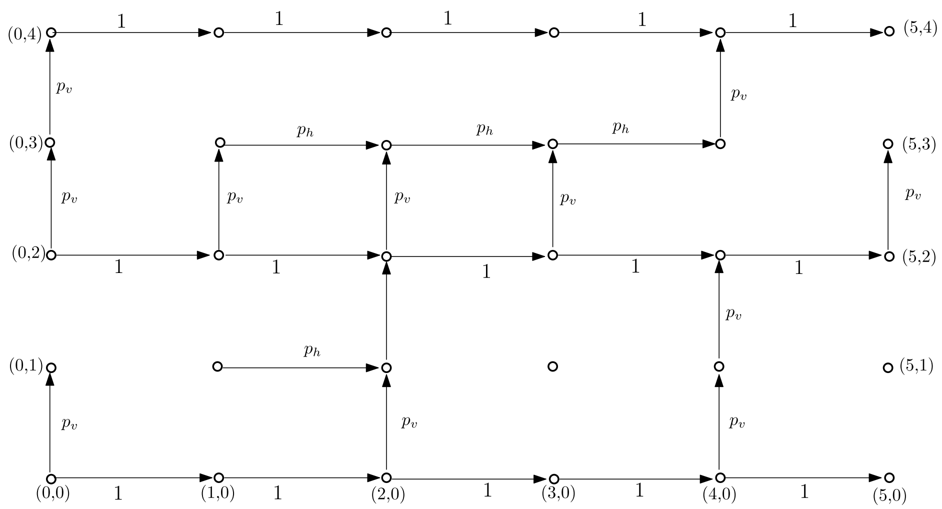

Fig. 1. A typical percolating configuration on a 5×4 lattice with hole $(3, 1)$. Open circles denote lattice sites. Oriented edges are occupied with probabilities shown. Empty edges carry probabilities $1 - p_h$ and $1 - p_v$ in the horizontal and vertical directions, respectively.

vertical edges are occupied with a probability p_v and whose horizontal edges in the n-th row are occupied with a probability 1 if n is odd and p_h if n is even. In particular, for $p_h = 0$ or 1, the model reduces to the Domany–Kinzel model, as shown below. However, the model is not a compact directed percolation class for $p_h \in (0, 1)$, since the connected cluster may has hole(s) (as shown in Fig. 1). Nevertheless, the hole(s) is (are) rectangular and its (their) width(s) is (are) 2;, it is believed that the critical behavior of the model refers to the compact directed percolation class and not to the habitual directed percolation class.

Given any $\alpha > 0$ and $p_h \in [0, 1]$, $p_v \in (0, 1)$, throughout this article, let $N_\alpha = \lfloor \alpha N \rfloor = \sup\{m \in \mathbb{Z}_+ : m \leq \alpha N\}$ with $N \in \mathbb{Z}_+$ and denote a two-dimensional rectangular net $\{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 0 \leq m \leq M \text{ and } 0 \leq n \leq N\}$ by $M \times N$. Let $\mathbb{P}$ be the probability distribution of the bond variables. Define the two-point correlation function as follows:

$$\tau(N_\alpha, N) = \mathbb{P}((N_\alpha, N) \text{ is percolating}).$$

It was shown by the method of steepest descent [9] that there is

$$\alpha_c = [1 - p_v^2 - p_h(1 - p_v)^2]/2p_v^2, \tag{1.1}$$

such that

$$\lim_{N\to\infty} \tau(N_{2\alpha}, 2N) = \begin{cases} 1 & \text{if } \alpha > \alpha_c, \\ 0 & \text{if } \alpha < \alpha_c, \\ \dfrac{1}{2} & \text{if } \alpha = \alpha_c. \end{cases} \tag{1.2}$$

For $\alpha < \alpha_c$, the critical exponent of the correlation length $\nu = 2$ is the same as that found in the Domany–Kinzel model [7,10–12].

The behavior of (1.2) is interesting, since the critical point is discontinuous. It is appropriate to define some of the standard critical exponents and to sketch the phenomenological scaling theory of $\tau(N_\alpha, N)$. For $\alpha < \alpha_c$, the scaling theory of critical behavior now asserts that the singular part of $\tau(N_\alpha, N)$ varies asymptotically as (see [13])

$$\tau(N_\alpha, N) \sim \frac{A_\alpha}{N^\eta} \exp\left(\frac{-B_\alpha N}{(\alpha_c - \alpha)^{-\nu}}\right), \tag{1.3}$$

where $f_1(N) \sim f_2(N)$ means that $\lim_{N\to\infty} f_1(N)/f_2(N) = 1$, the constants A_α and B_α depend on α, and $\eta, \nu \in (0, \infty)$ are universal constants. Furthermore, η is called the critical exponent and ν is called the critical exponent of the correlation length [14]. Note that there has been no general proof of the existence of critical exponents. To the best of our knowledge, the rate of convergence of (1.2) is unknown, and the values of A_α, B_α, and η in (1.3) for $\alpha \in (0, \alpha_c)$ are unknown too, even for Domany–Kinzel model. This allows us analyze (1.2) in detail.

Probability theory is a powerful tool to deal with this model. In fact, we can get α_c in (1.1) and the result of (1.2) by the law of large numbers rather than the method of steepest descent. Furthermore, the Berry–Esseen theorem attempts to quantify the rate at which this convergence to normality takes place. In this paper, we use the Berry–Esseen theorem to obtain sharp new results.

The rest of this paper is organized as follows. In Section 2, we state the main results (Theorems 2.1, 2.2 and 2.4) of this paper. Theorem 2.1 is proven in Section 3. In Section 4, we prove Theorem 2.2 by Theorem 2.1 and we apply Theorem 2.2 to show Theorem 2.4.