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Path probability of stochastic motion: A functional approach

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HIGHLIGHTS

- The concept of path probability of a random walker is studied.
- The general formula for the path probability distribution functional is derived.
- The overdamped limit of the formula is evaluated.
- The probability of finding paths inside a given tube is calculated.
- The theory developed here is applied to the problem of stock price in finance.

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ABSTRACT

The path probability of a particle undergoing stochastic motion is studied by the use of functional technique, and the general formula is derived for the path probability distribution functional. The probability of finding paths inside a tube/band, the center of which is stipulated by a given path, is analytically evaluated in a way analogous to continuous measurements in quantum mechanics. Then, the formalism developed here is applied to the stochastic dynamics of stock price in finance.

1. Introduction

Consider a certain stochastic equation for $X(t)$ that contains an external noise $\eta(t)$. A solution of the equation satisfying a certain initial condition, $X_\eta(t)$, defines a trajectory of a particle/walker. Then, it is essential to determine the probability of finding the particle in the interval $[x,\ x+\mathrm{d}x]$ at time t: $f(x,\ t)\ \mathrm{d}x$, where $f(x,\ t)$ is the probability distribution function given by $f(x,\ t) = \left\langle \delta\left(x - X_\eta(t)\right)\right\rangle_\eta$ with $\langle\bullet\rangle_\eta$ being the average over the noise (see the next section). The passage from the stochastic equation to the probability distribution function is commonly established through the Fokker–Planck equation [1,2]. This is a widely discussed issue that enables one to describe important phenomena such as diffusion and transport.

A less discussed issue may be *path probability*, $P[x]$. In this case, one is concerned with the probability distribution functional that the particle follows the path $x(t)$. One of the first works relevant to this problem may be that in Ref. [3], where irreversible relaxation processes to equilibria are studied for macroscopic thermodynamic variables. Recently, the problem of path probabilities has been revisited in Ref. [4]. It seems, however, that still some points regarding the action functional remain to be clarified in that work.

Our purpose here is to develop the theory of path probabilities in stochastic processes based on the functional method. First, we derive the general formula for the path probability distribution functional associated with the Langevin equation.

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Then, we examine the overdamped limit, in which the formula becomes simplified drastically. As a byproduct, at this stage, the statement about the action functional made in Ref. [4] is clarified and modified. To obtain a finite measure, we further consider the probability of finding a path inside a "tube", or a "band" in a single dimension, the center of which is prescribed by a given path. In particular, there we employ the Gaussian filtering method inspired by the concept of continuous measurements in quantum mechanics [5]. We discuss these issues in the application of the formula to the stochastic dynamics of stock price in finance.

2. Path probability

The path probability distribution functional is defined as follows:

$$P[x] = \left\langle \delta\left[x - X_\eta\right]\right\rangle_\eta, \tag{1}$$

where $\delta\left[x - X_\eta\right]$ is the delta functional given by

$$\delta\left[x - X_\eta\right] = \prod_t \delta\left(x(t) - X_\eta(t)\right) \tag{2}$$

with the symbol Π_t being the continuous infinite product. The normalization condition is expressed by the functional integral: $\int \mathcal{D}x\, P[x] = 1$, where $\mathcal{D}x$ is the functional integration measure $\mathcal{D}x \equiv \Pi_t\, \mathrm{d}x(t)$.

Suppose that $X_\eta(t)$ is a solution of the Langevin equation of a particle with unit mass

$$\frac{\mathrm{d}^2X(t)}{\mathrm{d}t^2} = -\kappa\frac{\mathrm{d}X(t)}{\mathrm{d}t} + F\left(X(t)\right) + \sqrt{2D}\,\eta(t) \tag{3}$$

defined in a time interval, $0 < t < T$. Here, κ and D are positive constants referred to as the friction and diffusion coefficients, respectively. $F\left(X(t)\right)$ is a deterministic force term, whereas $\eta(t)$ is the unbiased Gaussian white noise satisfying

$$\langle\eta(t)\rangle_\eta = 0, \qquad \left\langle\eta(t)\,\eta(t')\right\rangle_\eta = \delta(t - t'). \tag{4}$$

The diffusion coefficient is pulled out from the noise as in Eq. (3) for later convenience. Einstein's relation states that, in equilibrium, κD is Boltzmann's constant times the temperature of the environment, an influence of which is represented by the noise. The explicit form of the distribution of $\eta(t)$ is given by

$$p[\eta] = \mathcal{N}\,\exp\left[-\frac{1}{2}\int \mathrm{d}t\,\eta^2(t)\right], \tag{5}$$

where $\mathcal{N}$ is the normalization factor and this symbol will commonly be used in the subsequent discussion. The average of a certain quantity, $Q(\eta)$, over the noise is expressed as $\langle Q(\eta)\rangle_\eta = \int \mathcal{D}\eta\, Q(\eta)\, p[\eta]$, and with this Eq. (4) can immediately be ascertained. Actually, the normalization factor is formally divergent. There are some methods to regularize it. One possible way is to discretize the time parameter. This issue will be discussed later in an explicit example (see Section 5).

Let us calculate the path probability in Eq. (1) for a solution $X_\eta(t)$ of Eq. (3). For this purpose, we rewrite Eq. (3) as

$$\Phi(X) \equiv \frac{\mathrm{d}^2X(t)}{\mathrm{d}t^2} + \kappa\frac{\mathrm{d}X(t)}{\mathrm{d}t} - F\left(X(t)\right) - \sqrt{2D}\,\eta(t) = 0. \tag{6}$$

Then, holds the following equality:

$$\delta\left[\Phi(x)\right] = \left|\mathrm{Det}\left[\frac{\delta\Phi}{\delta x}\right]\right|^{-1}\delta\left[x - X_\eta\right]. \tag{7}$$

The factor in front of the delta functional on the right-hand side denotes the functional determinant given by

$$\mathrm{Det}\left[\frac{\delta\Phi\ (x(t))}{\delta x(t')}\right] = \mathrm{Det}_{t,t'}\left\{\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \kappa\frac{\mathrm{d}}{\mathrm{d}t} - F'\left(x(t)\right)\right]\delta(t - t')\right\}, \tag{8}$$

where $F'(x) = \mathrm{d}F(x)/\mathrm{d}x$. It is noted that the value of this quantity is not fixed until the temporal boundary condition is imposed on the solution of the Langevin equation. However, we do not specify such a condition explicitly here.

Substituting Eq. (7) into Eq. (1), we obtain the following general formula:

$$P[x] = \mathcal{N}\left|\mathrm{Det}\left[\frac{\delta\Phi}{\delta x}\right]\right|\exp\left\{-\frac{1}{4D}\int_0^T \mathrm{d}t\,\left[\ddot{x}(t) + \kappa\,\dot{x}(t) - F\left(x(t)\right)\right]^2\right\}, \tag{9}$$

where the overdots mean the differentiations with respect to time. This shows how the quantity in the exponential is radically different from the ordinary action functional in the Euclidean path integral approach to quantum mechanics, in contrast to the claim made in Ref. [4].