Physica A 451 (2016) 465–474

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Inequality measures in kinetic exchange models of wealth distributions

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HIGHLIGHTS

- Inequality in kinetic exchange models of wealth distributions.
- Lower and upper bounds of inequality indices.
- Phase diagram of Gini index and *k*-index.
- Analytical formulas for gamma and double gamma distributions to calculate inequalities.

ARTICLE INFO

Article history:
Received 11 September 2015
Received in revised form 4 January 2016
Available online 8 February 2016

Keywords:
Kinetic models of wealth distribution
Inequality
Gini index

ABSTRACT

In this paper, we study the inequality indices for some models of wealth exchange. We calculated Gini index and newly introduced *k*-index and compare the results with reported empirical data available for different countries. We have found lower and upper bounds for the indices and discuss the efficiencies of the models. Some exact analytical calculations are given for a few cases. We also exactly compute the quantities for Gamma and double Gamma distributions.

1. Introduction

Socio-economic inequality [1–4] is manifested in the existence of unequal rewards and opportunities for social positions or statuses in a society. Structured, recurrent patterns of unequal distributions of goods, wealth, opportunities, rewards and punishments are mainly measured in terms of *inequality of conditions*, and *inequality of opportunities*. The former refers to the unequal distribution of income, wealth and material goods, while the latter refers to the unequal distribution of 'life chances' of individuals. This is somehow reflected in measures such as level of education, health status, and treatment by the criminal justice system. Socio-economic inequality often results in crisis, political unrest and instability, conflict, war, criminal activity and finally affects economic growth [5]. Initially, economic inequalities were studied in the context of income and wealth [6–8], but the notions and observations have led to widespread research, see e.g. Refs. [9,10] for various socio-economic inequalities. The study of inequality in society [11–13] is a topic of global focus and utmost current interest, bringing together researchers from various disciplines.

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http://dx.doi.org/10.1016/j.physa.2016.01.081

By the end of the 19th century, Pareto [14] made extensive studies and found that wealth distribution in Europe follows a power law for the rich, commonly known to be the *Pareto law*. Subsequent studies have revealed that the distributions of income and wealth possess some globally robust features (see, e.g., Ref. [7]): the bulk of both the income and wealth distributions seem to reasonably fit both the log-normal and the Gamma distributions. Economists have a preference for the log-normal distribution [15,16], while statisticians [17] and physicists [18,19,6] root for the Gamma distribution for the probability density or Gibbs/exponential distribution for the corresponding cumulative distribution. The high end of the distribution, known as the 'tail', is well described by a power law as observed by Pareto. Formally, the probability distribution of wealth is given by

$$P(m) \sim \begin{cases} F(m) & \text{for } m < m_c, \\ \dfrac{\alpha m_c^{\nu}}{m^{1+\nu}} & \text{for } m \geq m_c, \end{cases} \tag{1}$$

where α is a constant and ν is called the Pareto exponent, ranging between 1 and 3 [7] (see Ref. [20], for a historical account of Pareto's data and some recent sources). $F(m)$ is some function which could be exponential, Gamma or lognormal. The crossover point m_c is extracted from the numerical fittings.

One of the key class of models uses the kinetic theory of gases [21], where the gas molecules colliding and exchanging energy was mapped to agents meeting to exchange wealth, following certain rules [19]. In these models, a pair of agents agree to trade, each save a fraction λ of their instantaneous money/wealth and exchanges a random fraction of the rest at each trading step. The distribution of wealth in the steady state, $P(m)$ matches well with the empirical data. When the saving fraction λ is fixed, i.e., in case of homogeneous agents (CC model hereafter) [22], $P(m)$ are very well approximated to Gamma distributions [23]. It is important to note that, in reality, the richest follow a different dynamic where heterogeneity plays the key role. To obtain the power law distribution of wealth for the richest, one needs simply to consider each agent as different in terms of the fraction of wealth he/she saves in each trading [24], which is very natural to assume, because it is quite likely that agents in a market think differently from one another. With this very little modification, one can explain the whole range of wealth distribution [19]. When λ is distributed uniformly in $[0, 1)$ and quenched, (CCM model hereafter), i.e., for heterogeneous agents, one obtains a Pareto law for the probability density of wealth $P(m) \sim m^{-\nu}$ with exponent $\nu = 2$ [24,19]. Several variants of these models, find possible applications in a variety of trading processes [7,25].

Socio-economic inequalities are quantified in various ways. The most popular measures are absolute, in terms of indices, e.g., Gini [16], Theil [26], Pietra [27] and the recently introduced k index [28]. The alternative approach is a relative measure, in terms of probability distributions of various quantities, but the most of the above mentioned indices can be computed from the distributions. Most quantities often display broad distributions, usually lognormals, power-laws or their combinations. For example, the distribution of income is usually an exponential followed by a power law [29] (see Ref. [7] for other examples).

To compute the Gini index, one has to consider the Lorenz curve [30], that represents the cumulative proportion X of ordered individuals (from lowest to highest) in terms of the cumulative proportion of their sizes Y (see Fig. 1(a)). X can represent income or wealth of individuals. The Gini index (g) is defined as the ratio between the area enclosed between the Lorenz curve and the equality line, to that below the equality line. If the area between (i) the equality line and the Lorenz curve is A, and (ii) that below the Lorenz curve as B, the Gini index is given by $g = A/(A + B)$. The recently introduced 'k index' [28] is defined as the fraction k such that $(1 - k)$ fraction of individuals possess k fraction of income or wealth (see Fig. 1(a)) [31].

In this paper, we investigate the inequality in wealth in some models of wealth distribution which are inspired by kinetic theory of gases. We mainly discuss the results for two well studied models (CC and CCM) and a new model for bimodal distribution of wealth. We numerically compute the inequality indices, Gini index and k-index to quantify the inequalities. Gini index g, the most popular and widely used measure for inequality in case of income and wealth distribution, can take value from 0 to 1. The value $g = 0$ refers to complete equality and $g = 1$ represents completely inequality. The meaning of k-index, say, for a wealth distribution, is the following: k fraction of the top wealthiest people possess $1 - k$ fraction of total wealth. We found that in both CC and CCM models there are some upper and lower limits of the indices. For CC model, g varies between 0 and 0.5, and k from 0.5 to 0.68. Similarly, for CCM model, g varies between 0.4 and 0.85. Therefore, both models independently do not cover the possible theoretical range of values of g and k. We find that the range of g as found from empirical data $(0.2 - 0.7)$ (see Fig. 1, using World Bank data [32]) can be well covered by CCM model. We also considered a model where two groups of agents have fixed but different saving propensities. Depending on the combinations, the resulting probability distribution of wealth is found to be unimodal or bimodal. The phase boundaries, depending on the ratio of the two groups and the combination of values of their saving propensities are also computed numerically. The bimodal distribution seems to fit well to a combination of two gamma distributions (double-Gamma distribution). Gini index and k index are calculated for this model for different combination of parameters. Next, we considered gamma and double-Gamma distribution and computed certain quantities like Lorenz curve and Gini indices.

2. Models and numerical simulation results

Kinetic exchange models of wealth distributions [19] serve as simple paradigmatic models for exchange of wealth in an economy. The main idea is that agents possess wealth m_i which is redistributed upon trading with others. The 'economy' is