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Numerically pricing American options under the generalized mixed fractional Brownian motion model

Wenting Chen [a,b,*], Bowen Yan [b], Guanghua Lian [c], Ying Zhang [d]

[a] School of Business, Jiangnan University, Wuxi City, Jiangsu 214100, China
[b] School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia
[c] School of Commerce, University of South Australia, SA, Australia
[d] Shanghai Branch, Small Enterprise Finance Department, China Citic Bank, Shanghai 200120, China

HIGHLIGHTS

- The pricing of American puts is considered under the GMFBM model.
- The PDE governing the prices of vanilla options under the GMFBM model is derived for the first time.
- A sharp error estimate is given for the proposed numerical scheme.

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ABSTRACT

In this paper, we introduce a robust numerical method, based on the upwind scheme, for the pricing of American puts under the generalized mixed fractional Brownian motion (GMFBM) model. By using portfolio analysis and applying the Wick–Itô formula, a partial differential equation (PDE) governing the prices of vanilla options under the GMFBM is successfully derived for the first time. Based on this, we formulate the pricing of American puts under the current model as a linear complementarity problem (LCP). Unlike the classical Black–Scholes (B–S) model or the generalized B–S model discussed in Cen and Le (2011), the newly obtained LCP under the GMFBM model is difficult to be solved accurately because of the numerical instability which results from the degeneration of the governing PDE as time approaches zero. To overcome this difficulty, a numerical approach based on the upwind scheme is adopted. It is shown that the coefficient matrix of the current method is an M-matrix, which ensures its stability in the maximum-norm sense. Remarkably, we have managed to provide a sharp theoretic error estimate for the current method, which is further verified numerically. The results of various numerical experiments also suggest that this new approach is quite accurate, and can be easily extended to price other types of financial derivatives with an American-style exercise feature under the GMFBM model.

1. Introduction

An American-style option is an option that can be exercised at any time up to its expiration date. In financial engineering, it is a well known fact that the pricing of American options has long been acknowledged as a much more "intriguing problem" [1], with the challenge stemming from its early exercise nature. Mathematically, this nature has casted the pricing

* Corresponding author.
E-mail addresses: wtchen@jiangnan.edu.cn (W. Chen), Andy.Lian@unisa.edu.au (G. Lian), zhangying3_sh@citicbank.com (Y. Zhang).

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problem into a linear complementarity problem (LCP) or a free boundary problem, which is highly nonlinear and far more difficult to deal with than its European counterpart. Nowadays, it is important to ensure that American-style securities can be priced accurately as well as efficiently because of the popularity of this kind of options in today's financial markets. For a variety of techniques designed for accurately pricing American-style options under the Black–Scholes (B–S) model, please refer to Refs. [2,3] and the references therein.

Recent empirical studies [4–7] have, however, suggested that the well-known B–S model fails to capture the long-range dependence of real asset returns. To incorporate this issue, the literature advocates the introduction of the so-called fractional Brownian motion (FBM). In addition to the ability of capturing the long-range dependence of the asset returns, the FBM is also able to produce a burstiness in its sample path, which is another important behavior of the time series of the asset returns [8]. It should be pointed out that the FBM is neither a Markov nor a semi-martingale [9]. As a result, the Wick product, instead of the classical Itô theory, is used to define the stochastic integral, and the FBM is wickbitrage free. However, as pointed out in Refs. [10,11], a wickbitrage-free model such as the FBM may in some cases still lead to implementable naive arbitrage opportunities. To resolve this issue while still taking into account the long-range dependence of the asset returns, the mixed fractional Brownian motion (MFBM) is introduced. This model is a linear combination of a Brownian motion (BM) and a FBM, and has been widely used in the option pricing filed by many authors [12–14].

Recently, Thäle [15] further generalized the MFBM to the generalized mixed fractional Brownian motion (GMFBM), which is a linear combination of a countable number of BMs and FBMs. This generalization has received quite a lot of attention since it came into existence, because the GMFBM model includes the BM, FBM and MFBM as its simplest cases. For example, Suryawan [16] studied the GMFBM in the white noise space, and derived explicit expressions for its S-transform and distributional derivative. Zanten [17] provided necessary and sufficient conditions for the equivalence between the GMFBM and a single FBM. Following his work, He and Chen [18] derived a sufficient condition for the market modeled under the GMFBM to be arbitrage free.

Although the properties of the GMFBM have been studied by many authors as stated above, the pricing of option derivatives under this model, especially from the partial differential equation (PDE) point of view, is still a new topic. A recent progress is made in Ref. [18] by deriving closed-form analytical solution for the price of credit default swaps. Their work is mainly based upon a statistical point of view; the PDE system governing the price of vanilla options under the GMFBM model remains unknown.

In this paper, we consider the pricing of American puts under the GMFBM model. By using the portfolio analysis and the Wick–Itô formula, we contribute to the literature the PDE governing the prices of options under the GMFBM model, based on which, the pricing of American puts is then formulated as a LCP. Unlike the classical B–S model or the generalized B–S model discussed in Ref. [19], the parabolic PDE will degenerate to a hyperbolic type when the time is approaching zero, as will be discussed in the later sections. This would result in numerical instability if those numerical methods proposed in Refs. [19–21] were still adopted to solve the option price under the current model. A robust numerical method which can solve for the American option prices effectively under the GMFBM model is the main contribution of the current work. We have shown theoretically that the coefficient matrix of our method is an M-matrix, which ensures the maximum-norm stability. Most remarkably, we have successfully obtained a sharp error estimate of the current scheme. It is shown that our method is first order convergent in both the time and spatial directions. Numerical results agree with the theoretical statement as well.

The rest of the paper is organized as follows. In Section 2, we derive the PDE governing the price of vanilla options under the GMFBM model, based on which, the pricing of American puts is formulated as a LCP. In Section 3, we introduce the numerical method in detail including its stability and error analysis. In Section 4, some numerical experiments are provided to test the theoretical error estimation obtained in the previous section. Concluding remarks are given in the last section.

2. American options under the GMFBM model

2.1. The GMFBM model

The concept of the GMFBM is recently introduced by Thäle [15] based upon the existing literature of FBM and MFBM. It is formally defined as follows.

Definition 2.1. A GMFBM of parameter $H = (H_1, \dots, H_N)$ and $\alpha = (\alpha_1, \dots, \alpha_N)$ is a stochastic process $Z^H = (Z_t^H)_{t\geq 0} = (Z_t^{H,\alpha})_{t\geq 0}$ defined on some probability space (Ω, F, P) as

$$Z^{H,\alpha} = \sum_{k=1}^{N} \alpha_k B_{H_k}(t),$$

where $(B_{H_k}(t))_{t\geq 0}$ are independent FBMs of Hurst parameters H_k, for $k = 1, \dots, N$ and $H_k \in (0, 1)$.

For a detailed survey on the properties of the GMFBM, we refer to Refs. [15,16] and the references therein.