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Scaling of weighted spectral distribution in deterministic scale-free networks

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HIGHLIGHTS

- A new deterministic model of scale-free networks is constructed.
- The WSD in the deterministic model grows sublinearly with network size.
- The WSD of a high-degree node provides a sensitive discrimination.
- The WSD depends on the connection relationships between small-degree nodes.
- We study the effects of local world, node deleting and assortativity adjustment.

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ABSTRACT

Scale-free networks are abundant in the real world. In this paper, we investigate the scaling properties of the weighted spectral distribution in several deterministic and stochastic models of evolving scale-free networks. First, we construct a new deterministic scale-free model whose node degrees have a unified format. Using graph structure features, we derive a precise formula for the spectral metric in this model. This formula verifies that the spectral metric grows sublinearly as network size (i.e., the number of nodes) grows. Additionally, the mathematical reasoning of the precise formula theoretically provides detailed explanations for this scaling property. Finally, we validate the scaling properties of the spectral metric using some stochastic models. The experimental results show that this scaling property can be retained regardless of local world, node deleting and assortativity adjustment.

1. Introduction

Many networks evolve over time. Specifically, because networks of different sizes may originate from the same evolving system, it is worthwhile to capture the scaling properties independent of network size. Scaling properties of graph metrics are widely studied in different types of networks, including real networks and stochastic (deterministic) models of networks. Lin et al. [1] and Jia et al. [2] analyzed the growth and evolution of topological features of the US airport network. Colman et al. [3] constructed a stochastic model of evolving complex networks that use local rewiring rules. Additionally,

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deterministic models of networks have also received significant attention [4–7], because they help us to understand the construction rules embedded in randomized structures and allow for analytic computation of additional graph metrics. The graph structure strongly affects the efficiency for random walks and a key metric is the mean first-passage time [8,9], that is, the expected first arriving time for the walker starting from a source node to a given target node. The average receiving time is defined as the average of mean first-passage times for a random walker to a given target hub node (averaged over all source points in the graph) [8,9]. Dai et al. [8] and Sun et al. [9] studied scaling properties of both the average weighted shortest path and the average receiving time in two deterministic scale-free network models, and they found that the two metrics are bounded or grow sublinearly with network size.

Graph spectra incorporate the eigenvalues of several graph matrices, including adjacency matrices, Laplacian matrices and normalized Laplacian matrices [10]. Cetinkaya et al. found that, in contrast to the spectral radii (i.e., the largest eigenvalues) of the adjacency and Laplacian spectra, the spectral radius of the normalized Laplacian spectrum is a good indicator of graph connectivity when comparing graphs of different sizes [11,12]. The natural connectivity (NC) defined on the adjacency spectrum represents the weighted sum of cycles of all lengths [13–17]. Wu et al. shown that, in contrast to the algebraic connectivity (i.e., the second smallest eigenvalue of the Laplacian spectrum), the NC in random graphs and regular ring lattices is asymptotically independent of network size [13–17]. The weighted spectral distribution (WSD) defined on the normalized Laplacian spectrum counts the number of N-cycles normalized by the degree of each node in the cycle, which represents the distribution of eigenvalues falling in $[0, 2]$ [18]. Fay et al. utilized the WSD to distinguish between different classes of graphs [19], including graphs from generators, internet application graphs and dK-random graphs.

Many networks of scientific interest are scale-free [4,20], that is, the probability randomly selecting a node with degree k decays as a power law, $P(k) \propto k^{-\gamma}$, where γ is the degree exponent. In this paper, we study the scaling properties of the WSD in several deterministic and stochastic models of evolving scale-free networks. Recent work on the WSD [18,19] has focused on the distribution of eigenvalues, but we investigate structural features of high-degree and small-degree nodes in evolving scale-free networks and apply those features to analyze scaling properties independent of network size. Our contributions are represented as follows: (i) based on the precise formula and the mathematical reasoning of a new deterministic scale-free network model, we theoretically investigate the scaling properties of the WSD in this model; and (ii) using several stochastic models with local world, node deleting and assortativity adjustment, we verify the scaling properties of the WSD experimentally.

2. Graphs and the WSD

$G = (V, E)$ denotes an undirected, simple graph where V and E are the node set and edge set, respectively, and d_v denotes the degree of a node v in graph G. As a result, the normalized Laplacian matrix of graph G is defined as follows [10]:

$$L(G)(u, v) = \begin{cases} 1 & \text{if } u = v \text{ and } d_v \neq 0 \\ -\dfrac{1}{\sqrt{d_u \cdot d_v}} & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The normalized Laplacian spectrum is comprised of all eigenvalues of the matrix $L(G)$: $0 = \lambda_1 \leq \cdots \leq \lambda_n \leq 2$. The WSD is a metric defined on the spectrum as follows [18]:

$$W(G, N) = \sum_{i=1,2,\ldots,n} (1 - \lambda_i)^N . \tag{2}$$

Calculating eigenvalues of a large (and even sparse) matrix is computationally expensive. Thus, Sylvester's Law of Inertia is utilized to calculate $f(\lambda = \theta)$ (i.e., the number of eigenvalues that fall in a given interval $\Omega_i \subseteq [0, 2]$ where $\theta \in \Omega_i$). Then, the WSD can be transformed into the following [18]:

$$W(G, N) \approx \sum_{\theta \in \Omega_i} (1 - \theta)^N \cdot f(\lambda = \theta) \tag{3}$$

where $\{\Omega_i\}_{i=1}^{h}$ are equally spaced intervals in $[0, 2]$. The number of equally spaced intervals h can be increased depending on the required precision.

Specifically, $\{f(\lambda = \theta)/n, \theta \in \Omega_i \mid i = 1, 2, \ldots, h\}$ represents the distribution of the normalized Laplacian spectrum where n is the number of nodes in graph G and the number of eigenvalues of the matrix $L(G)$, and the WSD basically is the weighted sum of the spectral distribution with a scaling factor n (the different weights are $(1 - \theta)^N$ where $\theta \in \Omega_i$). Therefore, the metric defined by Eqs. (2) and (3) is called an intuitive name (i.e., weighted spectral distribution) [18].

Additionally, the WSD counts the sum over all N-cycles in graph G [18]:

$$\sum_{i=1,2,\ldots,n} (1 - \lambda_i)^N = \sum_{C} \frac{1}{d_{u_1} d_{u_2} \ldots d_{u_N}} \tag{4}$$