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# An exactly solvable nonlinear model: Constructive effects of correlations between Gaussian noises

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**Abstract**

A system with two correlated Gaussian white noises is analysed. This system can describe both stochastic localization and long tails in the stationary distribution. Correlations between the noises can lead to a nonmonotonic behaviour of the variance as function of the intensity of one of the noises and to a stochastic resonance. A method for improving the transmission of external periodic signal by tuning parameters of the system discussed in this paper is proposed.


*Keywords:* Correlated Gaussian noises; Stochastic resonance

## 1. Introduction

Stochastic models with multiplicative, or parametrical, noise find numerous applications in a variety of branches of science and technology. Unfortunately, models for which analytical results are known are very scarce and any such a model deserves a thorough discussion. Recently, Denisov and Horsthemke [1] have discussed a model given by the equation

$$\dot{x} = -ax + |x|^{\alpha}\eta(t), \tag{1}$$

where $0 \leqslant \alpha \leqslant 1$, $\eta(t)$ is a Gaussian noise, possibly coloured, and have found that it can describe anomalous diffusion and stochastic localization. Denisov and Horsthemke have also discussed several physical systems in which models of type (1) can be useful; see references provided in their paper. Later Vitrenko [2] has generalized (1) to include two noise terms:

$$\dot{x} = -(a + \eta_1(t))x + |x|^{\alpha}\eta_2(t), \tag{2}$$

where $\eta_{1,2}$ are certain coloured and correlated Gaussian noises. This system has a very nice feature: for $0 < \alpha < 1$, it interpolates between a linear transmitter with multiplicative and additive noises ($\alpha = 0$) and a system that closely resembles a linear system with a purely multiplicative noise ($\alpha = 1$). These two linear

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systems are very well known in the literature (see e.g. Ref. [3] and references quoted therein). Vitrenko has formally linearized system (2) by means of a substitution that has been already used in Ref. [1]:

$$y = \frac{x}{|x|^{\alpha}}, \tag{3}$$

and solved the resulting equation for the trajectories. Converting back to the original variable proves to be rather tricky and that author has managed to do so only if the noises $\eta_{1,2}$ are correlated in a very specific (not to say peculiar) manner. It is now widely recognized that correlations between various noises can lead to many interesting effects. It is, however, possible that phenomena reported by Vitrenko result principally from the very specific form of correlations assumed by this author and are not generic to system (2). We find it interesting to see how the system behaves for the intermediate values of $\alpha$ when the correlation requirements are less restrictive than those discussed by Vitrenko.

Coloured noises introduce more complexity. However, if a dynamical effect is present in the white noise case, it also appears, perhaps in a distorted form, in the coloured case [4]. To simplify the discussion, we will assume that the noises are white. Finally, note that the expression $a + \eta_1(t)$ in Eq. (2) can be interpreted as a biased noise. The noise that multiplies $|x|^{\alpha}$ in Eq. (2) is not biased. To "symmetrize" the system, we include a bias in $\xi_2$ in our analysis. It is also convenient to have explicit expressions for noise amplitudes, or coupling constants between the noises and the dynamical variable. We thus recast Eq. (2) in the form

$$\dot{x} = -(a + p\xi_1(t))x + |x|^{\alpha}(b + q\xi_2(t)), \tag{4}$$

where $a > 0$, $0 \leqslant \alpha \leqslant 1$, $\xi_{1,2}$ are mutually correlated Gaussian white noises:

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t)\xi_i(t') \rangle = \delta(t - t'), \quad i = 1, 2, \tag{5a}$$

$$\langle \xi_1(t)\xi_2(t') \rangle = c\delta(t - t') \tag{5b}$$

and $c \in [-1, 1]$. If not otherwise specified, we interpret the noises in the sense of Ito. For the sake of terminology, we will call the noise $\xi_1(t)$ "multiplicative" and $\xi_2(t)$ "additive", even though this terminology is accurate only if $\alpha = 0$ (for $\alpha > 0$ both noises couple parametrically). Note that if a particle hits $x = 0$, it stays there forever if $\alpha > 0$. Accordingly, any fraction of the initial population that starts at $x = 0$ remains there and may be trivially excluded from the subsequent discussion of stationary distributions.

There is, in fact, one more reason for including $b \neq 0$ in our discussion. Much as substitution (3) linearizes system (4), another substitution, namely

$$z = \frac{|x|^{\alpha}}{x} \tag{6}$$

converts it to a noisy logistic equation

$$\dot{z} = (1 - \alpha)(a + p\xi_1(t))z - (1 - \alpha)(b + q\xi_2(t))z^2. \tag{7}$$

We have discussed this last system in Ref. [5,6] and found that $b \neq 0$ together with correlations between the noises can lead to a nonmonotonic behaviour of the variance $\langle z^2 \rangle - \langle z \rangle^2$ as a function of the intensity of the "additive" noise, $q$, and to a stochastic resonance [7] if the system is additionally stimulated by an external periodic signal. It would be naive to expect that these phenomena occur in system (4) in exactly the same manner as they do in (7). A nonlinear change of variables, especially in case of stochastic equations, can significantly alter the behaviour. We will see, however, that there are striking similarities between systems (7) and (4).

This paper is organized as follows: we construct the Fokker–Planck equation for system (4) in Section 2 and in Section 3 we present its stationary solutions. Then in Section 4 we discuss the constructive effects of the correlations between the noises; in particular, in Section 4.2 we give numerical evidence for the presence of the stochastic resonance. Conclusions are given in Section 5.