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A planar network between short and long range behaviour

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HIGHLIGHTS

- The Lucena Network is the dual of a multifractal partition of the square. The Projected Lucena Network **PLN** is the Lucena Network projected into the square lattice.
- By construction the **PLN** is a planar network that is embedded in the plane.
- The **PLN** follows the relation $P(r) \propto r^{-\delta}$ for $P(r)$ the probability of a node be connected to another node at distance r.
- The fractal dimension, d_f, of the **PLN** depends on its internal asymmetry, we identify two regimes: short range, $\delta \simeq 3.9$, and long range, $\delta \simeq 3.2$.
- For the short range limit $d_f \simeq 2$, the dimension of the embedding. For long range $d_f \simeq 2.5$.

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ABSTRACT

We study the Projection of Lucena Network **PLN**: a planar network whose nodes coincide with the sites of the square lattice. The **PLN** has one internal parameter ρ, for $\rho \to 1$ the network resembles a symmetric regular lattice with almost none highly connected node while $\rho \to 0$ is more asymmetric and there are several highly connected nodes. We estimate $P(r)$, the probability of a node be connected to another node at distance r. The **PLN** follows the fractal scaling $P(r) \propto r^{-\delta}$ with $3 < \delta < 4$ according to ρ. For $\rho \to 1$ we have δ close to 4 which is a signature of short range interactions, while $\rho \to 0$ shows $\delta \to 3$ and has long range interactions. In addition, the fractal dimension, d_f, of the **PLN** behaves as: for $\rho \to 1$, $d_f = 2$, the dimension of the embedding space while for $\rho \to 0$, d_f increases and the **PLN** shares some similarities with a small world graph.

1. Introduction

In the last decade the study of networks has been a fruitful research avenue [1,2], scientists realized that networks are useful for modelling a large class of physical, biological and social systems [3–5]. Despite the fact that network theory was originally formulated in a geometric free framework, many real networks are spatially embedded, for instance, airline networks [6], social networks [7] or human travel networks [8]. In this way several attempts to model spatially constrained networks have been proposed [9–11]. In this work instead of suggesting a new spatial embedded model we explore fractal characteristics of the planar Lucena network which is naturally a bidimensional structure.

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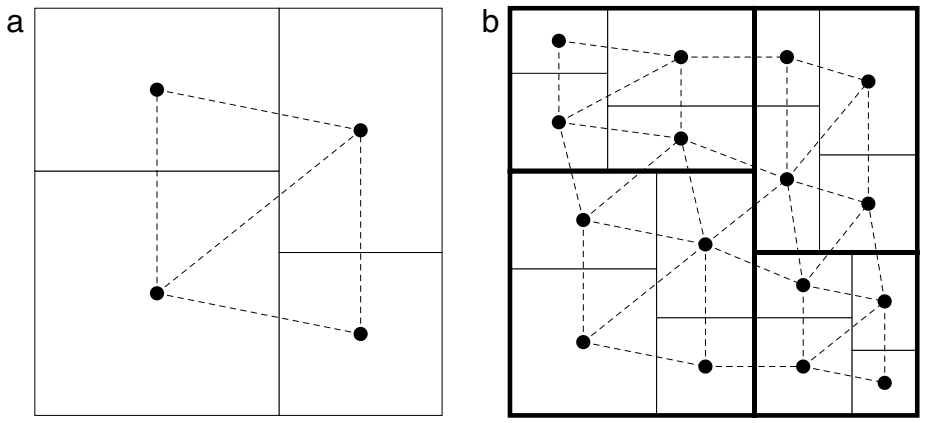

Fig. 1. Illustration of the multifractal lattice (lines) and Lucena network (dotted lines). The main pattern is shown by a square with vertical and horizontal sections following the parameter ρ. In figures (a) and (b) we show both the multifractal and the Lucena networks for $n = 1$ and $n = 2$ respectively. At each step the pattern shown in (a) is repeated inside all new rectangles. We use thick lines in (b) to highlight the original pattern. In the transition from n to $n + 1$ each vertex is erased and replaced by four new ones; the number and the topology of new connections at each step depend on ρ.

Before the analysis of Lucena network we discuss the Kosmidis, Havlin, Bunde model **KHB** [12] that has inspired much of our work, Ref. [13] also employed a similar model. The **KHB** model is an algorithm developed to study spatially embedded networks for one and two dimensions [14–16]. In the **KHB** model links are created randomly over nodes of a square lattice and the probability of linking nodes decays as a power-law with their distance. In a mathematical formulation we have:

$$P(r) \propto r^{-\delta} \tag{1}$$

for $P(r)$ the probability to have a connection between two nodes situated at distance r. The regime of small δ characterizes a network with connections extended to long distances while large δ defines the short range regime. The standard example of long range interactions is the Erdös–Rényi model while the short range is typically of a regular lattice with only local connections. We work in this paper with a special subset of the embedded networks that fulfil condition (1)—the Projected Lucena Network **PLN**. This network, besides following (1) is non crossing, or planar, but before we examine its properties we will present some of the story of the Lucena network.

The Lucena network starts in the paper [17] that introduces a peculiar multifractal partition of the square, that means, the area lattice distribution of the blocks of the partition follows a multifractal distribution. A review of the properties of multifractal lattice is found in Ref. [18]. The Lucena network is by construction a graph that uses the connectivities (neighbourhood) of the multifractal lattice elements; in this sense Lucena network is the dual of a multifractal lattice. In contrast to the **KHB** network the **PLN** is planar. Indeed, the links in the **KHB** network can cross each other while by construction **PLN** links never cross. In addition, the **PLN** follows a deterministic algorithm while the **KHB** network is essentially aleatory. The geometric characteristics of the **PLN** make this object an interesting candidate to integrated circuits design [19].

In this paper we study some fractal properties of the **PLN** model. Our intention is to study short and long range behaviour in the **PLN** model and its implications with fractal properties of the network. As we shall see the **PLN** is a transition model between these two limits. In Section 2 we present in some detail the multifractal lattice, the Lucena network and its projection into the square lattice. In Section 3 we show the main results of our manuscript: we compute δ of relation $P(r) \propto r^{-\delta}$ and estimate the fractal dimension d_f for several ρ, the internal parameter of the **PLN** model. Finally in Section 4 we discuss our results in the light of other spatial constraint networks of the literature.

2. The construction of the Projected Lucena Network

Initially we present the algorithm of construction of the Lucena multifractal. In the sequence we show the Lucena network which is the dual of the multifractal lattice and its projection into the square lattice. The algorithm of the multifractal lattice starts with a square and a section ratio $0 < \rho < 1$, a free parameter of the partition. The first step of the algorithm, $n = 1$, consists of two sections of the square: a vertical and a horizontal following the same ratio ρ. At the second step the same operation is repeated inside each one of the four blocks. In this way, at step n there are 2^{2n} tiles. A schematic picture of the algorithm is illustrated in Fig. 1 for $n = 1$ and 2. Refs. [20,18] show a detailed discussion about internal symmetries, algebraic properties and topology of the multifractal lattice. In Fig. 1(b) we notice that the patterns of Fig. 1(a) are repeated inside each one of the four blocks, however a rotational freedom in this operation is possible. The multifractal lattice can be deterministic and follow one specific rotation pattern or assume a random rotation in the interactive process of the algorithm. In this work we assume, otherwise stated, a constant rotational pattern in the algorithm. In contrast, Refs. [21,22] explore a full random version of this algorithm for an aleatory parameter ρ.