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Financial time series modeling using the Hurst exponent

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HIGHLIGHTS

- A method to detect long memory in the presence of structural breaks is proposed.
- Memory in absolute returns is found to increase when memory in actual returns grows.
- A model able to explain the memory in both absolute and actual returns is proposed.
- The new model outperforms the GARCH(1, 1) in terms of the Kolmogorov–Smirnov test.

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ABSTRACT

This study aims to enhance the understanding of logarithmic asset returns. In particular, more emphasis is given to the long memory property of financial returns, a well documented stylized fact. However, in the presence of structural breaks other studies suggest that statistical tools such as the AutoCorrelation Function (ACF) can wrongly indicate long memory. We propose an insensitive to structural breaks method to test for dependence between distant observations. Furthermore, a model which combines memory in returns and memory in absolute returns is developed in two stages. First return series are segmented with respect to changes in the volatility and then the two parameters of the model are estimated. To assess the capabilities of the model, historical prices of the Standard and Poor 500 Index (S&P500), Financial Time Stocks Exchange 100 Index (FTSE100), Deutsche Boerse Ag German Stock Index (DAX) and Crude Oil are used.[1] Given the estimated parameters and the volatility within each regime, 10000 vectors are generated and compared to the original data in terms of the Kolmogorov–Smirnov (K–S) statistical test. The obtained results suggest that long memory is present and provide evidence that the additional memory information captured by the model improves financial returns modeling.

1. Introduction

Modeling financial data (logarithmic returns) often faces two challenges, the ability to predict the rate of change in direction (i.e. how quickly positive changes turn to negative) and the magnitude of the change (i.e. the absolute value) [1]. In this work we provide a novel model that captures this information while being compatible with a number of characteristics of financial data, observed and manifested through empirical studies known as stylized facts. Memory properties of financial data in some cases cannot be explained by traditional time series models and modeling based on memory is a challenging task. To this end we employ the Hurst exponent [2], an extensively used tool in Econophysics, and the discrete fractional Gaussian noise used in Time Series Analysis.

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[1] The data are available at: https://uk.finance.yahoo.com/, and are adjusted for splits, dividends and distributions.

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1.1. Financial returns

Definition 1. The logarithmic returns r_t at time $t \geq 0$ of the price P_t of a financial asset over a period τ are defined as

$$r_t = \log(P_{t+\tau}) - \log(P_t) = \log\left(\frac{P_{t+\tau}}{P_t}\right). \tag{1.1}$$

In this work daily returns are of interest, by convention we set $\tau = 1$.

Definition 2. The variance σ^2 of a financial asset is a measure of the dispersion in the probability density of the returns r_t of the underlying asset. In fact, it is the conditional volatility σ_t^2 that is of interest which is obtained by the following expression

$$\sigma_t^2 = \text{Var}(r_t|F_{t-1}) \tag{1.2}$$

where F_{t-1} denotes the information set available at time $t-1$. Typically, F_{t-1} consists of all linear functions of the past returns.

The potential to predict stock market returns has fascinated market participants and researches for many decades. Nevertheless, forecasting financial returns still remains a difficult task. The first notable effort to probabilistically describe the evolution of financial asset prices was by Bachelier [3] in 1900. In his pioneering work Bachelier assumed that price changes are independent and of Gaussian nature. Empirical studies however have long recognized two major discrepancies between the Bachelier model and actual financial data. First, financial data commonly display temporal dependence in the alternation of periods of large price changes with periods of smaller changes. Second, the probability of extreme returns that are observed empirically is significantly higher than the probability of extreme returns under the normal distribution. To meet these challenges several new methods have been introduced including *AutoRegressive Conditional Heteroskedasticity* (ARCH) [4] and *Generalized AutoRegressive Conditional Heteroskedasticity* (GARCH) [5], which are the main reference models in literature. When GARCH models are entertained the return process and the conditional volatility are given by the following expressions

$$r_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \tag{1.3}$$

where $\{\epsilon_t\}$ is a sequence of i.i.d. random variables $\epsilon_t \sim N(0, 1)$. Also $\alpha_0 > 0, \alpha_i > 0, \beta_j > 0$ and $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$. The simplest case of the model is the GARCH(1, 1) where

$$r_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{1.4}$$

and $\alpha_0 > 0, \alpha_1 > 0, \beta_1 > 0$ and $(\alpha_1 + \beta_1) < 1$. The constraint $\alpha_1 + \beta_1 < 1$ implies that the unconditional variance of r_t is finite, while the conditional variance evolves over time.

Since the introduction of conditional volatility modeling several extensions of the GARCH model have been proposed aiming to capture more characteristics of the realized return series. The most notable examples are the IGARCH,GARCH-M, EGARCH and TGARCH models [6,7]. Nevertheless, the parsimonious GARCH(1, 1) compared to other more complex models it is often proved to perform equally well [8], which makes it the most popular model among practitioners. The impact of the aforementioned models has shaped the way of thinking in the analysis of time series of financial returns by introducing a "data driven" approach to statistical modeling, especially in finance. In particular, these methods were the first to comply with a number of common characteristics of returns observed and established through empirical studies known as "*Stylized Facts*".

1.2. Stylized facts

In the course of the past forty years or so, several independent studies have come to agree on a set of properties common across many instruments, markets and time periods [9–13,6]. The empirical results emerged from these studies are known as stylized facts. In this section several of these facts are observed and discussed using the historical values of the S&P500 index between June 1981 and March 2013. To better observe the behavior of the index a zoom in period from July-1987 to May-1988 is also available (see Fig. 1).

- **Volatility clustering**

Mandelbrot [14] analyzing the return series of financial assets observed that there exist volatility clusters. As in Fig. 2 there are time periods during which consecutive changes are large of either sign, as well as time periods for which small changes are followed by small changes.

- **Autocorrelations**

It has been observed that for any h (maybe apart from some small intraday time scales) the autocorrelations of asset returns $R_{r_t}(h) \approx 0$ where

$$R_{r_t}(h) = \frac{E[(r_t - \mu_{r_t})(r_{t-h} - \mu_{r_{t-h}})]}{\sigma_{r_t}\sigma_{r_{t+h}}}. \tag{1.5}$$