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The exclusion process: A paradigm for non-equilibrium behaviour

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HIGHLIGHTS

- Review of non-equilibrium processes (large-deviation, generalized detailed balance, Fluctuation Theorem).
- Basic properties of the Asymmetric Exclusion Process.
- Crash-course on the Bethe Ansatz.
- Exact results for large deviations of the stationary current.
- Towards a fluctuating hydrodynamics description.

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ABSTRACT

In these lectures, we shall present some remarkable results that have been obtained for systems far from equilibrium during the last two decades. We shall put a special emphasis on the concept of large deviation functions that provide us with a unified description of many physical situations. These functions are expected to play, for systems far from equilibrium, a role akin to that of the thermodynamic potentials. These concepts will be illustrated by exact solutions of the Asymmetric Exclusion Process, a paradigm for non-equilibrium statistical physics.

A system at mechanical and at thermal equilibrium obeys the principles of thermodynamics that are embodied in the laws of equilibrium statistical mechanics. The fundamental property is that a system, consisting of a huge number of microscopic degrees of freedom, can be described at equilibrium by only a few macroscopic parameters, called state variables. The values of these parameters can be determined by optimizing a potential function (such as the entropy, the free energy, the Gibbs free energy…) chosen according to the external constraints imposed upon the system. The connection between the macroscopic description and the microscopic scale is obtained through Boltzmann's formula (or one of its variants). Consider, for example, a system at thermal equilibrium with a reservoir at temperature T. Its thermodynamical properties are encoded by Boltzmann–Gibbs canonical law:

$$P_{\text{eq}}(\mathcal{C}) = \frac{e^{-E(\mathcal{C})/kT}}{Z}$$

where the Partition Function Z is related to the thermodynamic Free Energy F via

$$F = -kT \operatorname{Log} Z.$$

This expression (which is a consequence of Boltzmann's formula $s = k \log \Omega$) shows that the determination of the thermodynamic potentials can be expressed as a combinatorial (or counting) problem, which of course, can be extremely complex.

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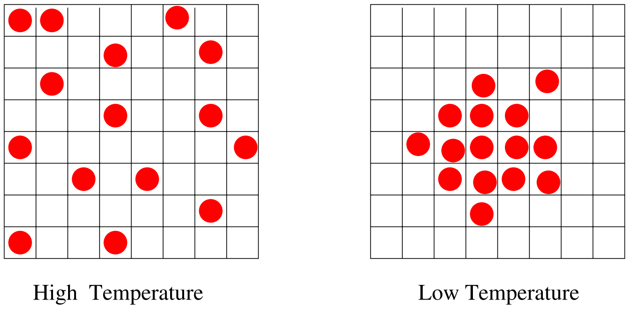

Fig. 1. The 2d Ising Model: the 'SCORE' of a given configuration is defined as the number of particles without a neighbour. Then, the probability to observe a configuration defined to be proportional to $e^{-\beta\,\mathrm{SCORE}}$, where β is proportional to the inverse temperature. This model displays a phase transition: At high temperature, $\beta \to 0$, the system does not display any order, we have gas; at low temperature, $\beta \to \infty$ a clustering occurs and the system is in a condensed phase. A phase transition occurs at a critical temperature β_c.

Nevertheless, equilibrium statistical physics provides us with a well-defined prescription to analyse thermodynamic systems: an explicit formula for the canonical law is given; this defines a probability measure on the configuration space of the system; statistical properties of observables (mean-values, fluctuations) can be calculated by performing averages with respect to this probability measure. The paradigm of equilibrium statistical physics is the Ising Model (see Fig. 1). It was solved in two dimensions by L. Onsager (1944). We emphasize that equilibrium statistical mechanics predicts macroscopic fluctuations (typically Gaussian) that are out of reach of Classical Thermodynamics: the paradigm of such fluctuations is the Brownian Motion.

For systems far from equilibrium, a theory that would generalize the formalism of equilibrium statistical mechanics to time-dependent processes is not yet available. However, although the theory is far from being complete, substantial progress has been made, particularly during the last twenty years. One line of research consists in exploring structural properties of non-equilibrium systems: this endeavour has led to celebrated results such as Fluctuation Theorems that generalize Einstein's fluctuation relation and linear response theory. Another strategy, inspired from the research devoted to the Ising model, is to gain insight into non-equilibrium physics from analytical studies and from exact solutions of some special models. In the field of non-equilibrium statistical mechanics, the Asymmetric Simple Exclusion Process (ASEP) is reaching the status of a paradigm.

In these lecture notes, we shall first review equilibrium properties in Section 1. Using Markov processes, we shall give a dynamical picture of equilibrium in Section 1.1. Then we shall introduce the detailed balance condition and explain how it is related to time reversal (Section 1.2). This will allow us to give a precise definition of the concept of 'equilibrium' from a dynamical point of view.

The study of non-equilibrium processes will begin in Section 2. We shall use as a leitmotiv for non-equilibrium, the picture of rod (or pipe) in contact with two reservoirs at different temperatures, or at different electrical (chemical) potentials (see Fig. 8). This simple picture will allow us to formulate some of the basic questions that have to be answered in order to understand non-equilibrium physics. The current theory of non-equilibrium processes requires the use of some mathematical tools, such as large-deviation functions, that are introduced, through various examples (Independent Bernoulli variables, random walk…), in Section 2.1; in particular, we explain how the thermodynamic Free Energy is connected to the large deviations of the density profile of a gas enclosed in a vessel. In Section 2.2, we show the relations between the large-deviation function and cumulants of a random variable. Section 2.3 is devoted to the very important concept of generalized detailed balance, a fundamental remnant of the time-reversal invariance of physics, that prevails even in situations far from equilibrium. Then, in Section 2.4, the Fluctuation Theorem is derived for Markov system that obey generalized detailed balance.

From Section 3 on, these lectures focus on the Asymmetric Exclusion Process (ASEP) and on some of the techniques developed in the last twenty years to derive exact solutions for this model and its variants. After a brief presentation of the model and of some of its simple properties (Sections 3.1–3.3), we apply the Mean-Field approximation to derive the hydrodynamic behaviour in Section 3.4; in particular, this technique is illustrated on the Lebowitz–Janowsky blockage model, a fascinating problem that has so far eluded an exact solution. Finally, in Section 3.5, the celebrated exact calculation of the steady state of the ASEP with open boundaries, using the Matrix Representation Method, is described.

Section 4 contains a crash-course on the Bethe Ansatz. We believe that the ASEP on a periodic ring, is the simplest model to learn how to apply this very important method. We try to explain the various steps that lead to the Bethe Equations in Section 4.1. These equations are analysed in the special TASEP case in Section 4.2.

We are now ready to calculate large deviation functions for non-equilibrium problems: this is the goal of Section 5. Our aim is to derive large deviations of the stationary current for the pipe picture, modelled by the ASEP. This is done first for the periodic case (ASEP on a ring) in Section 5.1, then for the open system in contact with two reservoirs (Section 5.2). The similarities between the two solutions are emphasized. Exact formulae for cumulants and for the large deviation functions are given. This section is the most advanced part of the course and represents the synthesis of the concepts and techniques that were developed in earlier sections. Detailed calculations are not given and can be found in recent research papers.