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A fractional mathematical approach to the distribution functions of quantum gases: Cosmic Microwave Background Radiation problem is revisited

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ABSTRACT

Efforts on the fundamentals of the nonextensive thermostatistical formulations of the realistic description of the physical systems have always been underway. In this context, the quantum systems of bosons and fermions are taken into consideration as g-ons. A new formalism of the unified distribution functions has been introduced using a fractional mathematical approach. With the purpose of verification of the theory, blackbody radiation problem has been investigated by making use of the generalized fractional Planck's distribution. In this context, the observed Cosmic Microwave Background Radiation (CMBR) energy density could be obtained exactly within nonextensive thermostatistical approach for the value $\alpha = 0.999983$ of the order of the fractional derivative and for the blackbody temperature $T = 2.72818$ K.

1. Introduction

The unaccountable behavior in classical physics of a physical system of fermions obeying Pauli exclusion principle could be explained by Fermi–Dirac (FD) statistics. The Fermi gas approximation which is accomplished by non-interacting independent fermions, could explain mainly the quantum mechanical properties of these systems. Similarly, quantum effects that are not accounted for in classical statistical mechanics are also observed experimentally in the bosonic systems. The Bose gas model which takes into account non-interacting independent bosons is successful in accounting for the quantum effects in these systems [1,2]. However, standard Bose–Einstein (BE) and FD statistics used in the frame of quantum statistics do not have competence in themselves to enlighten the realistic structure of nature. Therefore, for the sake of realistic description of the quantum systems, where the fractal structure of the space in which the system evolves and the long-range interaction between the particles are taken into account, generalized nonextensive thermostatistics is needed [3,4].

For this purpose, in the frame of nonextensive thermostatistics, one of the basic tools of thermostatistics namely the distribution function of the quantum gases is investigated where fractional mathematics is involved. In the course of this study, while taking into account the fractal structure of the space, the particles of the quantum gases i.e. bosons and fermions are unified as g-ons. Following Boltzmann's H-theorem, taking into account the entropy expression which has been generalized in accordance with the fractal structure of space for a system at equilibrium, the generalized distribution functions of the quantum gases are given by

$$\bar{n}_k(g,q) = \frac{1}{(2g-1) + \left[1+\beta(q-1)(\varepsilon_k-\mu)\right]^{1/(q-1)}} \tag{1}$$

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where, g is the statistical parameter taking values in the interval $0 \leq g \leq 1$, q is the entropy index, μ is the chemical potential and ε_k are the energies of the single particle states. For $g = 1$ the generalized FD and for $g = 0$ the generalized BE distributions are recovered. Since photons are belonging to the family of bosons, as a special case of Eq. (1), when $\mu = 0$ and $g = 0$ the generalized Planck's distribution

$$\bar{n}_k(q) = \frac{1}{[1 + (q-1)\beta\varepsilon_k]^{1/(q-1)} - 1} \tag{2}$$

is obtained. Standard FD, BE and Planck's distributions are recovered in the limit $q \to 1$ [5–7].

In the literature there are studies where dynamical behavior of a physical system could be described in a more realistic form by using fractional derivatives [8]. Particularly in mechanics, handling of some problems using fractional derivatives puts forward the relation between the order of the fractional derivative and fractal nature of the space [9,10].

In this study, fractional mathematics is used in the generalization of the distribution functions of the quantum gases. With this purpose, the fractional derivative operators are used in the variational calculations.

The content of the manuscript is organized as follows; in Section 2, a brief introduction of fractional calculus is presented. In Section 3, the fractional distribution functions of g-ons are obtained by making use of the Caputo fractional derivative operator and an investigation is carried out with the help of graphical representations. In Section 4, by employing one of the obtained distributions namely fractional Planck's distribution, the blackbody radiation problem is investigated in a fractional formalism and the results are interpreted with the help of plotted curves. In Section 5, with the help of the outcomes achieved, the nature of the deviation of CMBR from Planck's distribution is enlightened with fractional calculus approach. Moreover, it has been observed that; some of the implications of the fractal structure of space are straightforwardly exhibited in the fractional mathematical approach.

2. A short outline of fractional calculus

Fractional mathematics is a field of study in mathematical analysis where investigation and applications of ordinary (noninteger) order derivatives and integrals take place [8–17]. Recent studies on the subject exhibit the fact that; fractional mathematics are used in various fields of physics and engineering where systems evolving in fractal space are taken into account as well [18–21]. Nowadays, fractal geometry and fractional mathematics find applications in phenomenological theories for complex systems [22,23]. A vast number of problems which attract the attention of the physicists are studied using fractional mathematics [12–23]. Meanwhile, fractional approach is a mathematical instrument which puts forward the relation between the power law form of the systems in the nature and their deviation behavior from this form [24–26].

From theoretical point of view, fractional mathematics is based on the generalization of the integer order multiple integrals and integer order derivatives whose geometrical and physical interpretations are well known. In this content, a considerable number of fractional integral and derivative definitions are commonly used in the literature. Widely known among these definitions are the Riemann–Liouville (RL), Grünwald–Letnikov and Caputo definitions. In general, RL fractional derivative definition is used by mathematicians whereas Caputo fractional derivative is preferred by the physicists. There are two factors for the existence of such a distinction. These are; (i) the RL fractional derivative of a constant term is different from zero whereas in the Caputo fractional derivative it is identically zero, (ii) generally the initial conditions of the problems taken into account using RL fractional derivative are physically unacceptable. On the other hand, when the Caputo fractional derivative is introduced, one encounters physically reasonable initial conditions [12–14]. However, recently there exist promising efforts in the literature to provide physical interpretations for the fractional initial conditions [27].

According to the RL formalism, the fractional integral of order α, where α is any positive real number, is given by the definition:

$$J^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t > 0,\ \alpha \in R^{+}. \tag{3}$$

Caputo fractional derivative is defined by

$${}^{C}D^{\alpha} f(t) := J^{m-\alpha} D^{m} f(t) \tag{4}$$

where $\alpha > 0$. Here, m is the smallest integer greater than α i.e. $m-1 < \alpha < m$. In the special case where $\alpha = m$, the fractional derivatives transform to standard derivatives [12–14]. Caputo fractional derivative could be written in a more explicit form as follows:

$${}^{C}D^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\,\mathrm{d}\tau & m-1 < \alpha < m \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \alpha = m. \end{cases} \tag{5}$$

Caputo fractional derivative is more restrictive compared to RL fractional derivative since, the derivative of $f^{(m)}(\tau)$ has to be an integrable function. Moreover, definitions of RL and Caputo fractional derivatives are not equal to each other:

$$D^{m} J^{m-\alpha} f(t) \neq J^{m-\alpha} D^{m} f(t). \tag{6}$$