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A topological phase transition between small-worlds and fractal scaling in urban railway transportation networks?

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ABSTRACT

Fractal and small-worlds scaling laws are applied to study the growth of urban railway transportation networks using total length and total population as observational parameters. In spite of the variety of populations and urban structures, the variation of the total length of the railway network with the total population of conurbations follows similar patterns for large and middle metropolis. Diachronous analysis of data for urban transportation networks suggests that there is second-order phase transition from small-worlds behaviour to fractal scaling during their early stages of development.

1. Introduction

From the seminal work of Euler on the famous Königsberg bridges problem [1], graph theory has been increasingly developed and applied within a wide variety of scientific research fields. It is known that a number of geographical, biological and physical systems can be represented by networks constituted by a series of connected nodes. Many systems exhibit fractal geometries, characterized by structures that look the same on all length scales [2]. Fractal structures can be characterized by a scale law where the number of discrete units is proportional to the D-power of the size of those units, D being the fractal dimension of the structure. More recently, it has been obtained that, for a number of systems, an alternative model, the so-called small-worlds scaling, holds [3,4]. A network is considered a small-world if the average distance between nodes is at most a logarithm of the total system size [5] (see precise definitions in the next section).

Application of graph theory to urban structure, population distribution [6–10] and transportation systems [11–16] has received considerable attention. In the context of graph theory, different mathematical models have been developed in order to describe growing networks with geographical constraints [5,17–21]. Fractal and small-worlds are conceived as theoretical models whose suitability for the description of 'real' transportation networks has to be decided upon comparison of the predictions from such models with empirical data.

The determination of the topological properties of transportation systems involves, as emphasized by Lu and Tang [9], Kurant and Thiran [10], and Latora and Marchiori [13], a 'topological abstraction' where real, observable quantities are related to quantities defined by graph theories. For this purpose, different methods, namely, box-counting, node-counting, line-walk and length–area counts have been reported in the literature using available observational parameters [5,9,10,13]. As a result, several transportation systems have been characterized as fractal [9–12] and small-worlds [13–16] networks.

In this context, two general problems arise; by the first token, the possible coexistence of different scaling laws in transportation networks. Recently, Csányi and Szendröí [5] indicated that there is a clear dichotomy between fractal and small-worlds topologies in real-world networks, concluding that geographical constraints and degree correlations influence decisively the distinction between both types of scaling. Roughly, this dichotomy can be reformulated in terms of four questions: (a) is the social network of humans, *sensu stricto*, a small-world; (b) is the small-worlds behaviour confined to

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systems where a close vicinity exists between individuals; (c) does there exists an evolution in the scaling of real systems along time; (d) is there a possibility of mixed fractal plus small-worlds scaling behaviour or are these scaling responses mutually excluded.

A second general problem consists of the correlation between topological properties of transportation systems and observable geographical parameters provided by transportation models, such as the gravity one [22,23].

In the current report, a single approach, based on the relationship between the total length of a public transportation system and the total population of the conurbation, is described for relating the topologies of transportation networks and population. This approach is devoted to test the occurrence of fractal and small-worlds topologies in urban railway transportation systems, extending single modelling previously discussed [24]. As a result, the topology of contemporary metro/tramway systems of London, Madrid, Paris and Valencia is studied. The possible transition from small-worlds to fractal scaling during the evolution of such transportation networks along the 20th century is also examined. This formulation enables an examination of not only isochronous, but also diachronous analysis of transportation networks, providing a methodology for the comparative study of the time evolution of different transportation systems.

2. Operational approach for describing transportation networks

2.1. General models for transportation network growth

Let us consider a transportation network representative of a determined conurbation in a given 'historical' time. This is in general represented by a graph, a symbolic representation of the network and its connectivity. A network is then represented as a graph G formed by a collection of N points (nodes), every one connected to a finite number of nearest nodes by means of links. If two nodes are connected, they are labelled as neighbours. To analyze the topological structure of the network an idealized procedure is used by stepwise counting the number of nodes connected to a given starting node υ pertaining to the graph ($\upsilon \in G$) in successive node-to-node steps. Thus, in a first step, the node ν is connected to $N_\nu(1)$ 'contiguous' nodes, in a second step, to $N_\nu(2)$ nodes, etc. Then, the number of nodes of G which can be reached from the starting node υ in at most k steps ($k = 1, 2, \ldots$), $N_\nu(k)$, is defined as the size of the network at a given step k. The step number k in this process is labelled as the radius k neighbourhood of the node υ.

Let us consider a graph G with an infinite set of nodes. Following Csányi and Szendröi [5], fractal scaling can be represented by:

$$N_\upsilon(k) \propto k^D \tag{1}$$

where D represents the fractal dimension of the network. Small-worlds scaling satisfies:

$$N_\upsilon(k) \propto \mathrm{e}^{\alpha k} \tag{2}$$

where α is a small-worlds constant.

Testing of such theoretical relationships with observable data involves the determination of the $N_\upsilon(k)$ and k parameters at different steps during the growth of the network using the aforementioned node-counting, line-walk, etc. methods.

For finite networks G, the number of steps leading to the completion of the network is necessarily lower than a limiting value R. The size of neighbourhoods is often well approximated for $1 < k < R$ by a uniform scaling law, either (1) or (2), and at most a constant proportion of nodes lies outside this range. Under this assumption, the total number of nodes, N, is given by the relationships [5]:

$$N \propto \sum_{k=1}^{R} k^D \propto R^D \tag{3}$$

$$N \propto \sum_{k=1}^{R} \mathrm{e}^{\alpha k} \propto \mathrm{e}^{\alpha R} \tag{4}$$

for fractal and small-worlds scaling, respectively. Here, the k-summation proceeds from a given node by counting the nodes connected with the starting node in successive k-steps; i.e., constructing the entire framework from the starting node.

For such topologies, the average distance between υ and other nodes in the graph, l_υ, can be represented as:

$$l_\upsilon = \frac{1}{N}\sum_{k=1}^{R} k k^D \propto \frac{R^{D+1}}{N} \tag{5}$$

$$l_\upsilon = \frac{1}{N}\sum_{k=1}^{R} k \mathrm{e}^{\alpha k} \propto \frac{R\mathrm{e}^{\alpha R}}{N}. \tag{6}$$

As a result, fractal scaling implies:

$$l_\upsilon \propto N^{1/D} \tag{7}$$