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Dynamics of stochastic SEIS epidemic model with varying population size

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HIGHLIGHTS

- Stochasticity is introduced into an SEIS model with varying population size.
- Some simple conditions on extinction and persistent in the mean of the disease with probability one are shown.
- When the noises are small, there is a stationary distribution to stochastic model.

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ABSTRACT

We introduce the stochasticity into a deterministic model which has state variables susceptible–exposed–infected with varying population size in this paper. The infected individuals could return into susceptible compartment after recovering. We show that the stochastic model possesses a unique global solution under building up a suitable Lyapunov function and using generalized Itô's formula. The densities of the exposed and infected tend to extinction when some conditions are being valid. Moreover, the conditions of persistence to a global solution are derived when the parameters are subject to some simple criteria. The stochastic model admits a stationary distribution around the endemic equilibrium, which means that the disease will prevail. To check the validity of the main results, numerical simulations are demonstrated as end of this contribution.

1. Introduction

Some epidemic models assume that the populations have constant birth rate and mortality, such as Xiao and Ruan [1], Wei and Chen [2], Liu and Chen [3] and Zhao [4]. This assumption is somehow reasonable when diseases have limited effects on death rate and disappear in a short time. However, it fails to hold for diseases when human beings or animals population sizes are changing. In recent years, many authors had discussed corresponding epidemic models in a varying population environment, for instance, Refs. [5–11]. Amongst these work, Hui and Zhu [9] considered an SEIS epidemic model with varying population size. The system of differential equations is as follows:

$$\begin{aligned} \dot{S} &= (b-d)S + \gamma I - \lambda \frac{IS}{N}, \\ \dot{E} &= b(E+I) - dE + \lambda \frac{IS}{N} - \varepsilon E, \\ \dot{I} &= -(d+\alpha)I + \varepsilon E - \gamma I, \end{aligned} \tag{1}$$

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where $S(t), E(t), I(t)$ denote the densities of the susceptible, the exposed and infective individuals at time t respectively; b and d represent the natural birth rate and death rate; α is the rate related to death caused by disease; λ is an effective contact rate of infective individuals and $\lambda\frac{IS}{N}$ is a standard incidence rate; γ is the rate at which infective individuals recover and return to susceptible compartment; ε is the rate at which the exposed become infective individuals. All parameters in model (1) are assumed to be nonnegative and $b > 0$. The total population $N(t) = S(t)+E(t)+I(t)$ obeys scalar differential equation $\dot{N}(t) = (b-d)N(t) - \alpha I(t)$. By setting three new variables

$$x = \frac{S}{N}, \qquad y = \frac{E}{N}, \qquad z = \frac{I}{N}, \tag{2}$$

then, model (1) becomes the following system:

$$\begin{aligned} \dot{x} &= \gamma z - (\lambda - \alpha)xz, \\ \dot{y} &= bz + \alpha yz + \lambda xz - \varepsilon y, \\ \dot{z} &= \varepsilon y - (\alpha + \gamma + b)z + \alpha z^2. \end{aligned} \tag{3}$$

Hui and Zhu [9] studied the dynamical properties of system (3) in the domain $\Gamma = \{(x, y, z) \in \mathbb{R}^3_+ \mid x + y + z = 1\}$, and drew a conclusion that the infection-free equilibrium $P_0(1, 0, 0)$ of system (3) always exists. If $\lambda < \alpha + \gamma$, then $P_0(1, 0, 0)$ is globally asymptotically stable in Γ. If $\lambda > \alpha + \gamma$, $P_0(1, 0, 0)$ is unstable and there exists a unique endemic equilibrium $P^*(x^*, y^*, z^*)$, which is globally asymptotically stable when $\alpha \le \varepsilon$.

To make model (1) more realistic and reasonable, we introduce the corresponding stochastic environmental factor which is analogous to that of Imhof and Walcher [12]. Then, model (1) becomes

$$\begin{aligned} dS &= \left[(b-d)S + \gamma I - \lambda\frac{IS}{N}\right]dt + \sigma_1 S dB_1(t), \\ dE &= \left[b(E+I) - dE + \lambda\frac{IS}{N} - \varepsilon E\right]dt + \sigma_2 E dB_2(t), \\ dI &= [-(d+\alpha)I + \varepsilon E - \gamma I]dt + \sigma_3 I dB_3(t), \end{aligned} \tag{4}$$

where $B_i(t)$ $(i = 1, 2, 3)$ are independent Brownian motions, σ_i $(i = 1, 2, 3)$ are the intensities of the white noises. The equation of the total population size N is obtained from (4) and written as:

$$\dot{N}(t) = [(b-d)N - \alpha I]dt + \sigma_1 S dB_1(t) + \sigma_2 E dB_2(t) + \sigma_3 I dB_3(t).$$

We rewrite system (4) with three proportions x, y, z and have the following model:

$$\begin{aligned} dx \;&=\; [\gamma z - (\lambda-\alpha)xz - \sigma_1^2x^2 + x(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2)]dt \\ &\quad + \sigma_1 x(1-x)dB_1(t) - \sigma_2 xy dB_2(t) - \sigma_3 xz dB_3(t), \\ dy \;&=\; [bz + \alpha yz + \lambda xz - \varepsilon y - \sigma_2^2y^2 + y(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2)]dt \\ &\quad - \sigma_1 xy dB_1(t) + \sigma_2 y(1-y)dB_2(t) - \sigma_3 yz dB_3(t), \\ dz \;&=\; [\varepsilon y - (\alpha+\gamma+b)z + \alpha z^2 - \sigma_3^2z^2 + z(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2)]dt \\ &\quad - \sigma_1 xz dB_1(t) - \sigma_2 yz dB_2(t) + \sigma_3 z(1-z)dB_3(t). \end{aligned} \tag{5}$$

Using the relationship $x + y + z = 1$, we can omit analysis of the first equation of system (5), and discuss system (6) as follows:

$$\begin{aligned} dy \;&=\; [bz + \alpha yz + \lambda xz - \varepsilon y - \sigma_2^2y^2 + y(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2)]dt \\ &\quad - \sigma_1 xy dB_1(t) + \sigma_2 y(1-y)dB_2(t) - \sigma_3 yz dB_3(t), \\ dz \;&=\; [\varepsilon y - (\alpha+\gamma+b)z + \alpha z^2 - \sigma_3^2z^2 + z(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2)]dt \\ &\quad - \sigma_1 xz dB_1(t) - \sigma_2 yz dB_2(t) + \sigma_3 z(1-z)dB_3(t), \end{aligned} \tag{6}$$

with its given initial value $(y(0), z(0)) \in \mathbb{R}^2_+$ and $y(0) + z(0) < 1$. Let the valid domain be $\Gamma' = \{(y, z) \in \mathbb{R}^2_+ \mid y + z < 1\}$ and the complete probability space be $(\Omega, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions, that is, it is a right continuous function while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

2. Noise suppresses explosion of a global solution and extinction

Motivated by the proof of Theorem 2.1 in Ref. [13], according to the similar progress of proof, we demonstrate the definition of the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \min\{y(t), z(t)\} \le \frac{1}{k} \text{ or } \max\{y(t), z(t)\} \ge k\right\}$$