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Diffusion of innovations dynamics, biological growth and catenary function

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HIGHLIGHTS

- We propose an entropic connection between the catenary function and the aggregate logistic model.
- We find a linkage between logistic and Bass models for diffusion of innovations in social systems.
- Generalized Bass models determine perturbed catenaries through a control function.
- These entropic correspondences suggest a physical connection between static and dynamic equilibria.
- This invariance may be motivated by the Verlinde's conjecture on the entropic origin of gravity.

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ABSTRACT

The catenary function has a well-known role in determining the shape of chains and cables supported at their ends under the force of gravity. This enables design using a specific static equilibrium over space. Its reflected version, the catenary arch, allows the construction of bridges and arches exploiting the dual equilibrium property under uniform compression. In this paper, we emphasize a further connection with well-known aggregate biological growth models over time and the related diffusion of innovation key paradigms (e.g., logistic and Bass distributions over time) that determine self-sustaining evolutionary growth dynamics in naturalistic and socio-economic contexts. Moreover, we prove that the 'local entropy function', related to a logistic distribution, is a catenary and vice versa. This special invariance may be explained, at a deeper level, through the Verlinde's conjecture on the origin of gravity as an effect of the entropic force.

1. Introduction

Shannon's entropy is a basic conceptual and physical tool to understand statistical properties of systems. In statistics, it is used for prediction purposes in different areas in order to represent the natural variability of a population (system) with reference to qualitative and quantitative factors or to describe its decomposition, in multivariate contexts, highlighting some relationships (dependence among factors) and their relative strength. The basic reference is a distribution of probabilities $p_i,\ \ i = 1, 2, \ldots, K$, among K possible alternative states of a system, and there is no need to introduce space or time environments. This special feature is extremely relevant in the sequel.

The *local entropy* $\log(1/p_i)$ – based on a particular information unit of measure, a log concave function of the ratio $1/p_i$ – depicts a level of rarity of event i within the possible system states and, therefore, the level of information to detect or represent it. In other words, a state of a system, as an event with small probability, generates a high level of local entropy

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and, equivalently, information for its characterization is very high. Vice versa, events with high probability require small levels of information for their representation and, similarly, determine a low level of local entropy contribution within the system.

Shannon's entropy H is a mean value of *local entropies*, $\log(1/p_i)$, namely, $H = \sum_{i=1}^{K} p_i \log(1/p_i)$ for $p_i > 0$ and $\sum_{i=1}^{K} p_i = 1$. In an unconstrained system, we notice two polar frameworks: a zero entropy situation, $H = 0$, corresponding to a degenerate distribution, where only one event or state is possible, for instance event i with probability $p_i = 1$ and $p_j = 0$ for $j \neq i$, or, conversely, a maximal entropy situation, $H = \log K$, under the most uninformative distribution, $p_i = 1/K;\ i = 1, 2, \ldots, K$.

Entropy index H, or its normalized version $S = H/\log K$, denotes a measure of variability of a distribution of events stemming from a system. Nevertheless, it does not explain why those events are related to a specific system.

We may consider a further interpretation of *local entropy* in a system. It measures a hypothetical *attraction force* that the system exerts in maintaining the event i as a member of it. Index H denotes the mean-field attraction expressed within the current system. Under a degenerate distribution, we have a coherence among different interpretations: The variability is absent (zero), the rarity of the only possible event is minimal, $\log(1/p_i) = 0$, and the attraction force is absent (zero). Conversely, for small positive values of p_i, the contribution to variability is high, the rarity of the specific event is relevant and the attraction force to be a member of the system is significant. An equilibrium with maximal mean entropic attraction H is obtained under the uniform distribution $p_i = 1/K$ in an unconstrained environment. If the cardinality K of a homogeneous system (uniform distribution) augments, then an increasing entropy due to expansion of $\log K$ confirms that the maximal entropy is a monotone function of complexity.

In constrained environments over space or time, the distributions of events that define an observable equilibrium may be different from the uniform hypothesis. Correspondingly, the related function of local entropies is not invariant, $\log(1/p_i) \neq \log K$. Notice that with the term 'equilibrium' we denote special distributions, or their monotonic transformations, obtained under complex frameworks that may be represented, not exclusively, through equations and related initial conditions.

In this paper, we examine two different experiments that exhibit two equilibria, a first one over space and a second one over time. The proposed static equilibrium over space refers to the shape of a homogeneous chain supported at its ends, under the force of gravity. It is well known that the mathematical description of the previous form is a catenary function.

An apparently different situation is the aggregate dynamic expansion of a viral agent in a human population over time. In this case, under regularity conditions in a homogeneous population, the pertinent mathematical description is a probability density, a logistic distribution of events Verhulst [1] that depicts the time when we observe the change of state of individuals.

Both examples may have similar or analogous alternatives. In the first case, for instance, the catenary function properly defines, by reflection, the catenary arch under local uniform compression. In the second one, we may consider many other situations characterized by diffusion of innovations in a socio-economic context or growth models in naturalistic frameworks that may be conceived as variants of logistic distribution; for instance, the Bass models (see in particular Bass [2] and Bass et al. [3]).

The main aim of this paper is to prove that the dynamic equilibrium of a logistic distribution over time may be linked to a corresponding *local entropy function*, which is a catenary. In this way, the local attraction of each event in the system over time or the reciprocal contagion rate over time has a common representation with the static equilibria of suspended chains or catenary arches.

Conversely, starting from a static catenary, we prove that the local cardinality of the chain links, for unitary variations over space, defines, under the gravity force, a logistic distribution through a *local entropy function*.

We highlight that both examples refer to specific constraints. The chain has rigid and homogeneous links; it is not free but suspended at two separate ends. In the contagion or diffusion experiment, the susceptible population is homogeneous and limited. In both cases, the role of entropy and the 'attraction force' that it implies are the common bricks. The proposed connection, between static and dynamic equilibria in these two reduced but general cases, is in agreement with Verlinde's conjecture on the origin of gravity as an effect of entropic force (see Verlinde [4]).

A further issue of interest for both static and dynamic contexts, previously reduced to a common interpretative key, is the treatment of systematic deviations from uniformity or homogeneity assumptions. The generalized Bass model, GBM (see Bass et al. [3]), and the related perturbed logistic model may be easily converted to perturbed catenaries in order to take into account different intervention functions in controlling dynamics of viral or diffusion of innovation expansions or, in the static domain, the presence of local non-uniform compression due to design constraints. In this perspective, the proposed solution, through the local entropy function of a perturbed logistic or a GBM, simplifies the operative construction of weighted catenaries as characterized in Osserman [5].

The paper is organized as follows. Section 2 presents the catenary function, some historical aspects, applications in architecture and its recent weighted extensions by Osserman [5]. Section 3 introduces basic growth models and a diffusion of innovations perspective focusing on logistic and Bass models and their isomorphism. Section 4 establishes the fundamental result of this paper: the connection between a static equilibrium framework, described through a catenary, and the corresponding dynamic equilibria typical of logistic and related models in growth or diffusion of innovations contexts. In particular, Section 4.4 proposes a physical interpretation of the correspondence between the above-mentioned equilibria drawing on Verlinde's conjecture on gravity conceived as an entropic force. Section 5 introduces a more tractable definition of weighted catenaries via the generalized logistic model. In Section 6, we propose our conclusions.