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The q-gamma and (q,q)-polygamma functions of Tsallis statistics

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ABSTRACT

An axiomatic definition is given for the q-gamma function $\Gamma_q(x), q \in \mathbb{R}, q > 0, x \in \mathbb{R}$ of Tsallis (non-extensive) statistical physics, the continuous analogue of the q-factorial of Suyari [H. Suyari, Physica A 368 (1) (2006) 63], and the q-analogue of the gamma function $\Gamma(x)$ of Euler and Gauss. A working definition in closed form, based on the Hurwitz and Riemann zeta functions (including their analytic continuations), is shown to satisfy this definition. Several relations involving the q-gamma and other functions are obtained. The (q,q)-polygamma functions $\psi_{q,q}^{(m)}(x), m \in \mathbb{N}$, defined by successive derivatives of $\ln_q \Gamma_q(x)$, where $\ln_q a = (1-q)^{-1}(a^{1-q}-1), a > 0$ is the q-logarithmic function, are also reported. The new functions are used to calculate the inferred probabilities and multipliers for Tsallis systems with finite numbers of particles $N \ll \infty$.

1. Introduction

In the past two decades, a theory of non-extensive statistical mechanics has been developed based on the discrete Tsallis entropy function [1]:

$$H_q = -\sum_{i=1}^{s} p_i \ln_{2-q} p_i = \frac{1}{q-1}\left(1 - \sum_{i=1}^{s} p_i^q\right) \tag{1}$$

where p_i is the probability of occurrence of the ith distinguishable state or category, from a total of s such states; $q \in \mathbb{R}$ is the Tsallis parameter; and $\ln_q x = (1-q)^{-1}(x^{1-q}-1), x > 0$ is the q-logarithmic function [2]. (All q-algebraic terms are defined in more detail in Section 2.1.) In the limit $q \to 1$, $\ln_q x \to \ln x$ and H_q reduces to the traditional Boltzmann, Gibbs and Shannon ("BGS") entropy function [3–7]:

$$H = -\sum_{i=1}^{s} p_i \ln p_i. \tag{2}$$

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Tsallis statistical mechanics has been shown to be applicable to the analysis of systems with "long-range interactions" between entities within the system, including a wide assortment of physical, chemical, engineering, financial, social and other systems [8–12]. It also possesses several interesting statistical and mathematical properties [13–16].

In the combinatorial definition of entropy originally due to Boltzmann [3] and Planck [5] [see Refs. [17–23]], the BGS entropy function arises from the natural logarithm of the multinomial weight, subject to the approximation of Stirling [24] and de Moivre [25], $\lim_{a\to\infty} \ln a! \cong a\ln a - a$. In Tsallis statistical mechanics, it has recently been shown that the Tsallis entropy can be obtained from the q-logarithm of the q-multinomial weight, subject to the q-Stirling approximation (these terms are explained shortly) [26,27]. Tsallis statistical mechanics can therefore be derived by a q-mathematical distortion of the multinomial combinatorial structure usually adopted for statistical analysis. Recently, the so-called "non-asymptotic" form of the Tsallis entropy function was derived using this q-generalised Boltzmann principle, by taking a (modified) q-logarithm of the q-multinomial weight, without the q-Stirling approximation [27]. The analysis yields a q-generalisation of the non-asymptotic entropy function reported recently [17,18,22], as a function of q and the number of entities N. For mathematical reasons, this analysis is most conveniently presented using the q-gamma function $\Gamma_q(x) = \Gamma(q, x)$, $q \in \mathbb{R}$, $x \in \mathbb{R}$, the continuous analogue of the q-factorial $n!_q$, $q \in \mathbb{R}$, $n \in \mathbb{N}$ introduced recently by one author [26], and the q-analogue of the gamma function $\Gamma(x)$ of Euler and Gauss. (In the Tsallis mathematical literature, subscript q does not imply $q \in \mathbb{N}$ or $q \in \mathbb{Z}$; in most cases $q \in \mathbb{R}$.)

The aim of this study is to report working definitions of $\Gamma_q(x)$, and the derivatives of its q-logarithm, in closed form. This work is set out as follows. Firstly, a background on the q-algebra operations of Tsallis statistics and various other functions is provided, and some difficulties with the current "cutoff condition" employed in Tsallis mathematics are resolved. The behaviour of $\ln_q(n!_q)$ is then examined, leading to several theorems, the most important of which gives $\ln_q(n!_q)$ as a function of the Hurwitz and Riemann zeta functions; this is proved for $q \in \mathbb{R}, q > 2$ and $q \in \mathbb{Z}, q \leq 0$, and conjectured for $q \in \mathbb{R}, q < 2, q \notin \mathbb{Z}$, thus covering the domain $q \in \mathbb{R}, q \neq 1, 2$. An axiomatic definition for $\ln_q \Gamma_q(x)$ — analogous to the Bohr–Mollerup [28] definition of the gamma function — is then proposed, based on three mathematical axioms: an initial condition $\ln_q \Gamma_q(1) = 0, \forall q$, a recursive condition with respect to x, and the convexity of $\ln_q \Gamma_q(x)$ with respect to x; this is then used to define $\Gamma_q(x)$. A limit form of $\Gamma_q(x)$, a q-analogue of the Euler–Gauss limit expression for $\Gamma(x)$, is first examined; this is found to satisfy the above axioms but is not in closed form. However, by extension of a relation for $\ln_q(n!_q)$ in terms of zeta functions, a closed-form expression for $\ln_q \Gamma_q(x)$ — and hence for $\Gamma_q(x)$ — is obtained; this is shown to satisfy the three axioms and a number of auxiliary conditions. Several functional forms of $\Gamma_q(x)$ and $\ln_q \Gamma_q(x)$, and of the successive derivatives of $\ln_q \Gamma_q(x)$ — here termed (q, q)-polygamma functions — are also reported. The new functions are used to calculate the inferred probabilities and multipliers for various "non-asymptotic" Tsallis systems over a range of values of q, containing finite numbers of particles $N \ll \infty$.

2. Mathematical background

2.1. q-algebraic terminology

The analysis makes use of the following q-algebra (unless stated $q \in \mathbb{R}$, $N \in \mathbb{N}$, $n_i \in \mathbb{N}$, $\forall i$, $a \in \mathbb{N}$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$; note $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). In the following, many functions contain the cutoff condition:

$$[b]_+^{1/(1-q)} = \begin{cases} b^{1/(1-q)} & \text{if } b > 0 \\ \text{undefined} & \text{if } b \leq 0. \end{cases} \tag{3}$$

Note that this condition is a modification of that used previously in the Tsallis literature, in which the function is assigned zero for $b \leq 0$ [30,32]. This modification has been made to ensure that inverse functions within the Tsallis framework are bijective, i.e. for the function $\phi : X \to Y$, there exists a corresponding inverse function $\phi^{-1} : Y \to X$ such that $\phi^{-1}(\phi(x)) = x$ for all $x \in \{b(x) > 0\}$ (and vice versa). The following functions relate to Tsallis statistics [1,2,8–10,29–32], and differ from similar terminology used in the mathematics of quantum groups [33,34].

- q-exponential function [2,29,30]:

$$\exp_q(x) := [1 + (1-q)\, x]_+^{\frac{1}{1-q}}. \tag{4}$$

- q-logarithm function [2,29,30]:

$$\ln_q x := \frac{x^{1-q} - 1}{1-q}, \quad \text{if } x > 0. \tag{5}$$

- q-product [31,32]:

$$x \otimes_q y := \left[x^{1-q} + y^{1-q} - 1\right]_+^{\frac{1}{1-q}}, \ \text{if } x > 0, \quad y > 0 \tag{6}$$